Representing other operators in the Fock space.

Fields:

Define \( \hat{\Psi}(x) = \sum_k \Psi_k(x) a_k \)

\( \hat{\Psi}(x) \) \( \hat{a}_k \) is chosen single particle wave function (plane waves, gluons etc)

\( C \)-numbers

\( \hat{\Psi}^+(x) = \sum_k \Psi_k^+(x) a_k^+ \)

Remember \( x = \vec{x}, s_z, \text{isospin etc} \)

\( \Psi(x) \) is called field since it is an operator defined at all space points.

If we want to pull spin explicitly out we could write

\( \hat{\Psi}(x) = \begin{pmatrix} \Psi(x, \uparrow) \\ \Psi(x, \downarrow) \end{pmatrix} = \Psi(x, s_z) \)
Using the CCR we can show that
\[ \sum_k \Psi_k(x, \alpha) \Psi_k^*(x', \alpha') = \delta(x-x') \delta(\alpha - \alpha') \]
for completeness of single particle wave functions.

\[ \int \Psi(x, \alpha) \Psi^*(x', \alpha') \, d^2x \, d^2\alpha = 0 \]

What is the physical meaning of these operators?
First note that an operation like
\[ \langle x_1, x_2, \ldots, x_N | n_1, n_2, \ldots, n_m \rangle \]
does not make sense since \( |n_1, \ldots, n_m \rangle \) and \( |x_1, x_2, \ldots, x_N \rangle \) are vectors in two different Hilbert spaces while the notation \( \langle A | B \rangle \) means take \( \langle A | j(1A, 1B) \rangle \) i.e.
scalar product of vectors \( |A\rangle \) and \( |B\rangle \)
from the same space.
But what we can do is for example

$$\langle 0 | \hat{N}(x) | 1_k \rangle \rightarrow \text{one particle in state } k$$

$$= \langle 0 | \sum_k \psi_k(x) a_k a_k^\dagger | 10 \rangle$$

$$= \langle 0 | \sum_k \psi_k(x) [\sqrt{\delta_{k, k}} a_k a_k^\dagger - a_k^\dagger a_k] | 10 \rangle$$

$$= \sum_k \psi_k(x) \delta_{k, k} \langle 0 | 10 \rangle = \psi_k(x)$$

Thus $\psi_k(x)$ contains information about single particle orbitals. In general:

$$\phi_{n_1, n_2, \ldots, n_N}(x_1, x_2, \ldots, x_N) = \frac{1}{\sqrt{N!}} \langle 0 | \hat{\Psi}(x_1) \ldots \hat{\Psi}(x_N) | n_1, n_2, \ldots, n_N \rangle$$

$$= \frac{1}{\sqrt{N!}} \sum_{n_1, n_2, \ldots, n_N} \psi_{n_1}(x_1) \ldots \psi_{n_N}(x_N) \langle 0 | a_{n_1}^\dagger \ldots a_{n_N}^\dagger a_{n_1} \ldots a_{n_N} | 10 \rangle$$

$$\Rightarrow \sum_{i=1}^N n_i = N \Rightarrow m_i \neq 0$$

Move all $a$ to the right of $a^\dagger$ produces an integral to 0:

= turns into determinant $a^\dagger (s10 + e)$

Another minus sign

= turns into determinant $a^\dagger (s10 + e)$
It is also instructive to write the second quantized Hamiltonian in terms of field operators replacing $a, a^+$'s.

We had $\hat{H} = \hat{H}(a, a^\dagger)$.

Now $\hat{\Psi}(x) = \sum_k \Psi_k(x) a^\dagger_k$ can be thought of as a linear mapping from $a \to \Psi$.

The inverse is given by:

$$\int dx \hat{\Psi}^*(x) \Psi_k(x) = \sum_k \int dx \Psi_k(x) \Psi_k^*(x) a_k$$

operator $\hat{a}_k$ is a orthogonal of singlet pair wavefunction

$$\hat{a}_k = a_k$$

and similarly $\int dx \hat{\Psi}^+(x) \Psi_k(x) = a^\dagger_k$

Put it into $\hat{H}(a, a^\dagger)$ and get

$$\hat{H} = \int dx \hat{\Psi}^+(x) \hat{T}(x) \hat{\Psi}(x) + \frac{1}{2} \int dx dx' \hat{\Psi}^+(x) \hat{\Psi}^+(x') V(x, x') \hat{\Psi}(x') \hat{\Psi}(x)$$

first quantization form.

role...
In particular, for kinetic term:

\[ K = \sum_{s} \langle \psi_{s}\gamma s \rangle \langle \gamma s | 1 | T | s \rangle \]

\[ \langle \gamma s | 1 | \gamma s \rangle = \int d\mathbf{x}' d\mathbf{x} \langle \gamma s | \gamma s \rangle \langle \gamma s | \gamma s \rangle \]

\[ = \int d\mathbf{x}' d\mathbf{x} \langle \gamma s \rangle [\left(-i \frac{\partial}{\partial x}\right)^{2} \delta(x-x')] \langle \gamma s \rangle \]

\[ K = \sum_{s} \left( \int dy \langle \gamma s | \gamma s \rangle \right) \int d\mathbf{x}' d\mathbf{x} \langle \gamma s \rangle \left[\left(-i \frac{\partial}{\partial x}\right)^{2} \delta(x-x')\right] \langle \gamma s \rangle \]

\[ = \sum_{s} \left( \int dy \langle \gamma s | \gamma s \rangle \right) \int d\mathbf{x}' d\mathbf{x} \langle \gamma s | \gamma s \rangle \left[\left(-i \frac{\partial}{\partial x}\right)^{2} \delta(x-x')\right] \langle \gamma s \rangle \]

\[ \sum_{s} \langle \gamma s | \gamma s \rangle = 1 \]

\[ \Rightarrow K = \sum_{s} \left( \int dy \langle \gamma s | \gamma s \rangle \right) \int d\mathbf{x}' d\mathbf{x} \langle \gamma s | \gamma s \rangle \left[\left(-i \frac{\partial}{\partial x}\right)^{2} \delta(x-x')\right] \langle \gamma s \rangle \]

\[ K = \int d\mathbf{x} d\mathbf{x}' \langle \gamma s | \gamma s \rangle \left[\left(-i \frac{\partial}{\partial x}\right)^{2} \delta(x-x')\right] \langle \gamma s \rangle \]

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We can extend this to other operators.

Consider a general one-body operator
(i.e. acts on a single particle at the time,
unlike \( V(x, x') \) which needs two particles)

\[ J(x_i) \rightarrow \text{first quantization} \]

\[ J(x_i) \delta(x_i - x_i') = \langle x_i | J | x_i' \rangle \quad (\star) \]

Then one can most show that

\[ \frac{N}{Z} \sum_{i=1}^{N} J(x_i) \rightarrow \sum_{\nu} \langle \nu | J | \nu \rangle a^+_\nu a_\nu = \int dx \psi_\nu^+(x) J(x) \psi_\nu(x) \]

\[ \approx J \]

i.e. condide a state \( | 1_k \rangle \rightarrow \) one particle in level \( k \)

\( \langle 1_k | J | 1_k \rangle \) this would be the same as taking \( \langle x \rangle \nabla \)

The first quantized expression could evaluating it

in its paricular state \( (1_k \neq 1_x) \)

\[ \langle x| \nu \rangle = \psi_\nu(x) \quad \langle x' | \nu \rangle = \delta(x-x') \]
Check:

First quantization:

\[
\langle k | i j k | l l k \rangle = \int d^3 x \int d^3 x' \langle k | x \rangle \langle x | i j \rangle \langle i j | x' \rangle \langle x' | l l k \rangle
\]

\[
= \int d^3 x \int d^3 x' \; \psi^*_k(x) \mathcal{G}(x) \delta(x-x') \psi_j(x') = \int d^3 x \; \psi^*_k(x) \mathcal{G}(x) \psi_j(x)
\]

Second quantization:

\[
\langle i k | j j l k \rangle = \langle 0 | a^+_k \sum_{r,s} \langle r | l j l s \rangle a^+_r a^+_s a^+_l 10 \rangle
\]

\[
= \sum_{r,s} \langle r | l j l s \rangle \langle 0 | (a_k a^+_r - a^+_k a_r) (a^+_s a^+_l - a^+_l a^+_s) 10 \rangle \xrightarrow{-\epsilon} 0
\]

\[
= \sum_{r,s} \langle r | l j l s \rangle \delta_{kr} \delta_{sr} \langle 0 | 10 \rangle = \langle k | i j k | l l k \rangle \quad \square
\]
Another example: number operator.

First quantization: \( \hat{n}(y) \) asks the question if particle is at position \( y \):
\[
\langle x' | \hat{n}(y) | x \rangle = n(y) \delta(x' - x)
\]
\[
n(y) = \delta(y - x)
\]

Second quantization:
\[
\hat{n}(y) = \hat{\Psi}^+(y) \Psi(y) = \sum_{r,s} \hat{\psi}^*_r(y) \psi_s(y) \hat{a}^+_r \hat{a}_s
\]

Total number operator \( \hat{N} = \int dy \hat{n}(y) \)
\[
= \int dy \hat{\Psi}^+(y) \Psi(y) = \sum_r \hat{a}^+_r \hat{a}_r
\]

\[
\langle \psi_{k'}(y) | \hat{n}(y) | \psi_k(y) \rangle = \langle \psi_{k'}(y) | \sum_{r,s} \hat{\psi}^*_r(y) \psi_s(y) \hat{a}^+_r \hat{a}_s | \psi_k(y) \rangle
\]
\[
= \psi_{k'}^*(y) \psi_k(y) \delta(x' - x)
\]

In first quantization:
\[
\langle \psi_{k'}(y) | \hat{n}(y) | \psi_k(y) \rangle = \int dx' dx \langle \psi_{k'}(x') | \hat{n}(y) | \psi_k(x) \rangle
\]
\[
= \psi_{k'}^*(x') \psi_k(x) \delta(x' - x)
\]
\[
= \delta(x' - x) \psi_{k'}^*(x') \psi_k(x)
\]

Note that \( \langle \hat{N} \rangle = 0 \) implies total number of particles is conserved but total wave function contains an arbitrary number of particles.