The exam is a closed-book examination. You may not refer to lecture notes, textbooks, or any other course materials. You may use a calculator, but solely for the purpose of arithmetic computation. A list of potentially useful formulas, definitions and relations is given after the problems.

The exam will consist of five multiple-choice questions, each worth 2 points for a total of 10 points, and two problems to be worked out, each worth 10 points. For the two problems, you must show all work: clearly state and justify any arguments, assumptions, and approximations, as well as the use of any formulas. Unless otherwise specified, evaluate all integrals and derivatives, and perform any arithmetic calculations if a numerical answer is requested.

Multiple Choice Questions
(Choose only one answer and label your answer clearly.)

1. An electric field is most directly related to:
   (a) the momentum of a test charge
   (b) the kinetic energy of a test charge
   (c) the potential energy of a test charge
   (d) the force acting on a test charge
   (e) the charge carried by a test charge

2. Electric field lines:
   (a) are trajectories of a test charge
   (b) cross each other in the region between two point charges
   (c) indicate the direction of the electric field
   (d) form closed loops that start and end on a point charge
   (e) are none of the above

3. To make an uncharged object have a positive charge:
   (a) remove some neutrons
   (b) add some neutrons
   (c) add some electrons
   (d) remove some electrons
   (e) connect it to ground

4. Choose the correct statement:
   (a) A proton tends to go from a region of low potential to a region of high potential.
   (b) The potential of a negatively charged conductor must be negative relative to zero potential at infinity.
   (c) If \( \vec{E} = 0 \) at a point \( P \), then \( V \) must be zero at \( P \).
   (d) If \( V = 0 \) at a point \( P \), then \( \vec{E} \) must be zero at \( P \).
   (e) none of the above.

\[
\vec{F} = q \vec{E} = q \vec{E} \text{ due to all other charges at the position of } q_0
\]
5. If 500 J of work are required to carry a charged particle between two points with a potential difference of 20 V, the magnitude of the charge on the particle is

(a) 0.04 C
(b) 12.5 C
(c) 25 C
(d) can not be computed unless the path is given.
(e) none of the above

6. Three possible configurations for an electron e and a proton p are shown below. Take the zero of potential to be at infinity and rank the three configurations according to the potential at S from most negative to most positive.

(a) 1, 2, 3
(b) 3, 2, 1
(c) 2, 3, 1
(d) 1 and 2 tie, then 3
(e) 1 and 3 tie, then 2

7. A positively charged insulating rod is brought close to an object that is suspended by a string. If the object is attracted toward the rod, we can unambiguously conclude one of the following:

(a) the object is positively charged
(b) the object is negatively charged
(c) the object is an insulator
(d) the object is a conductor
(e) we do not have enough information to conclude any one of the above

8. A proton p and an electron e are on the x axis. The directions of the electric field at points 1, 2 and 3 respectively is given by:

(a) →, ←, →
(b) ←, →, ←
(c) ←, ←, ←
(d) ←, ←, →
(e) none of the above

9. Two uncharged metal spheres, L and M, are in contact. A negatively charged rod is brought close to L, but not touching it. The two spheres are slightly separated and the rod is then withdrawn. As a result:

(a) both spheres are neutral
(b) both spheres are positive
(c) both spheres are negative
(d) L is negative and M is positive
(e) L is positive and M is negative

10. If the electric potential is given by \( V(x, y) = C_1 x + C_2 y \), where \( C_1 \) and \( C_2 \) are constants, which of the following statements is NOT true?

(a) The electric field has components in the x and y directions only
(b) The electric field is uniform (constant in magnitude and direction everywhere).
(c) \( E_x = -C_1 \), \( E_y = -C_2 \)
(d) The potential difference, \( (V_B - V_A) \), between two points A at \( (x = 0, y = 0) \) and B at \( (x = d, y = 0) \) is \( C_2 d \).

11. An electric dipole in an external electric field has the least potential energy when it

(a) is parallel to the electric field
(b) anti-parallel to the electric field
(c) perpendicular to the electric field
(d) experiences no net external force
(e) none of the above
Problems

12. In the figure, two small objects carrying net charges $+Q$ and $-Q$ are shown, where $Q$ is a positive number with units of Coulombs. They are located on the y-axis, separated by a distance $6d$ meters, as shown. Suppose we place a third object with charge $+2Q$ at the point $P$ given by $(x = 4d, y = 3d)$.

What are the horizontal and vertical components of the electric force felt by this third object, expressed in terms of $Q$, $d$, and $\epsilon_0$?

Label charges as $Q_1$ and $Q_2$; find the electric force on charge at $P$ due to each of these two charges separately.

$$y = 6d$$

$$P(4d, 3d)$$

$\theta$

$x = 4d$

$\theta$

$\theta$

$P_1, Q_1 = 2Q, \theta \ell$

$P_2, Q_2 = 2Q, \phi \ell$

$F_{Q_1}$ due to $Q_2 = F_1$

$F_{Q_2}$ due to $Q_1 = F_2$

$\Rightarrow F_{Q_1} = F_1 + F_2$

Further, $F_{Q_1} = \frac{k|Q_1||Q_2|}{r^2} = \frac{2kQ^2}{25d^2}$

$F_{Q_2} = \frac{k|Q_1||Q_2|}{r^2} = \frac{2kQ^2}{25d^2}$

Look at vector addition $F_{Q_1}$ and $F_{Q_2}$:

$\left( F_{x_1}, F_{y_1}, F_{z_1}, F_{x_2}, F_{y_2}, F_{z_2} > 0 \right)$

From geometry, we have:

$F_x = F_{x_1} + F_{x_2} = 0$ and $F_y = F_{y_1} + F_{y_2} = 0$

Since $|F_{Q_1}| = |F_{Q_2}|$, $F_{Q_1} = F_{Q_2}$ and $F_{Q_1} = F_{Q_2}$. Therefore,

$F_{Q_1} = F_{Q_2} = \left( F_{x_1} = F_{x_1}, F_{y_1} = F_{y_1}, F_{z_1} = F_{z_1} \right) = -2F_{x_2}$

$= -2 |F_{Q_2}| \sin \theta \hat{j}$

Now, $\sin \theta = \frac{3d}{5d} = \frac{3}{5}$; hence

$F_{Q_1} = -2 \left( \frac{2kQ^2}{25d^2} \right) \frac{3}{5} \hat{j} = -\frac{12kQ^2}{125d^2} \hat{j}$
10. Find the electric field at P due to semicircular charged ring by direct integration.

\[ \text{(a) Linear charge density \( \lambda_p \):} \]
\[ \lambda_p = \frac{+Q}{\frac{1}{2} (2\pi R)} = \frac{Q}{\pi R} \]
\[ \lambda_{\text{bottom}} = -\frac{-Q}{\frac{1}{2} (2\pi R)} = -\frac{Q}{\pi R} \]

\[ \text{(b) Label the infinitesimal element of charge in the top half at \( \Theta \) w.r.t. y-axis as (1). The electric field due to this (small) charge element at P will be:} \]
\[ E_{\rho} = \frac{k}{R^2} \frac{d\rho}{R} \]
\[ E_{\phi} = \frac{k}{R^2} \frac{d\phi}{R} \]

Note: This is a differential element of electric field at P because it is due to a differential element of charge \( dq \).

11. Label the second infinitesimal element of charge in the bottom half at \( \Theta \) w.r.t. x-axis as (2). We have:

\[ \left| dE_p \right| = \frac{k}{R^2} \left| dq \right| = \frac{k}{R^2} dq \]

Vectors \( dE_x \) and \( dE_y \) (always originating at the point where we are interested in the \( E \)-field) are drawn in the figure. The magnitudes of these vectors are the same, and given the geometry, their \( x \)-components cancel.

\[ \left( dE_{x1}, dE_{x2}, dE_{x3}, dE_{x4}, dE_{x5} \right) \]
\[ dE_x = \left| dE_p \right| \sin \Theta \]
\[ dE_y = \left| dE_p \right| \cos \Theta \]
\[ dE_{x1} = \left| dE_p \right| \sin \Theta \]
\[ dE_{y1} = \left| dE_p \right| \cos \Theta \]

\[ dE_p = dE_x + dE_y \]

\[ \int \left( dE_{x2} \right) + \int \left( dE_{x3} \right) = -2 \int dE_y \]
\[ = -2 \frac{k}{R^2} \cos \Theta \]

12. The total \( E \)-field at \( P \) is obtained as a sum, or integral, over such charged pairs from \( \Theta = 0 \) to \( \Theta = \pi \) (covering the entire semicircle):

\[ E_P = \int dE_p = -E_P \]

\[ E_P = \left| \int dE_p \right| = \frac{2k}{R^2} \int \cos \Theta \, dq \]
\[ = \frac{2k}{R^2} \int_0^{\pi} \cos \Theta \cdot 2Q \, d\Theta \]
\[ = \frac{2Q}{R^2} \int \cos \Theta \, d\Theta \]
\[ = \frac{2Q}{R^2} \left[ -\sin \Theta \right]_0^{\pi} \]
\[ = \frac{2Q}{R^2} \left( -\sin \pi + \sin 0 \right) \]
\[ = \frac{2Q}{R^2} \cdot 0 \]
\[ = 0 \]

\[ = \frac{2Q}{R^2} \cdot 0 \]
\[ = \frac{2Q}{R^2} \cdot 0 \]
\[ = \frac{2Q}{R^2} \cdot 0 \]
\[ = \frac{2Q}{R^2} \cdot 0 \]
14. In the figure, a non-conducting rod of length $L$ has charge $q$ uniformly distributed along its length. Find the magnitude and direction of the electric field at $P$, which is located at a perpendicular distance $a$ from one end of the rod, as shown.

\[ \frac{dq}{dx} = \lambda \frac{dx}{L} \]

Total charge, $q$

\[ \int \frac{udv}{(u^2 + a^2)^{3/2}} = -\frac{k}{(u^2 + a^2)^{1/2}} + C \]
\[ \int \frac{du}{(u^2 + a^2)^{3/2}} = \frac{u}{\sqrt{u^2 + a^2}} + C \]

You may find the following integrals to be useful:

Consider contribution to the electric field at $P$ due to an element $dq$ located at $x$:

\[ |dE| = k \frac{dq}{r^2} = \frac{k}{(L-x)^2 + a^2} \]

\[ dE_x = |dE| \sin \theta = \frac{k}{(L-x)^2 + a^2} \frac{L-x}{\sqrt{(L-x)^2 + a^2}} \]

\[ dE_y = |dE| \cos \theta = \frac{k}{(L-x)^2 + a^2} \frac{a}{\sqrt{(L-x)^2 + a^2}} \]

Since there is no cancellation of $x$- or $y$-components $E_x$, we integrate both:

\[ E_x = \int_{red} dE_x = \int_{red} \frac{k(L-x)}{(L-x)^2 + a^2} \frac{dx}{(L-x)^2 + a^2} \]
Convert to an integral over $x$ by using:

$$d\phi = \left( \frac{Q}{L} \right) \frac{dx}{\sqrt{(L-x)^2 + a^2}}$$

Therefore:

$$E_x = \frac{kQ}{L} \left[ \frac{(L-x)\,dx}{\sqrt{(L-x)^2 + a^2}} \right]_0^L = \frac{kQ}{L} \left[ \frac{1}{\sqrt{\left( \frac{L-x}{a} \right)^2 + 1}} \right]_0^L$$

Use\[ \int \frac{dx}{(x^2 + a^2)^{3/2}} = -\frac{1}{(x^2 + a^2)^{1/2}} \]

Therefore:

$$E_x = -\frac{kQ}{L} \left[ \frac{1}{\sqrt{(L-x)^2 + a^2}} \right]_0^L = \frac{kQ}{L} \left( \frac{1}{a} - \frac{1}{\sqrt{L^2 + a^2}} \right)$$

since $u = \frac{L-x}{a}$, $du = -dx$

Now find the $y$-component:

$$E_y = \int_{rod} dE_y = \int_{rod} \frac{kQ\,dx}{\sqrt{(L-x)^2 + a^2}}$$

Use\[ \int \frac{dx}{(x^2 + a^2)^{3/2}} = \frac{1}{a^2 \sqrt{x^2 + a^2}} \]

Therefore:

$$E_y = -\frac{kQ}{L} \left[ \frac{L-x}{a^2 \sqrt{(L-x)^2 + a^2}} \right]_0^L = -\frac{kQ}{L} \left( \frac{L}{a^2 \sqrt{L^2 + a^2}} - 0 \right) = \frac{kQ}{aL \sqrt{L^2 + a^2}}$$

So:

$$\vec{E}_p = E_x \hat{i} - E_y \hat{j}$$

$$= -\frac{kQ}{L} \left( \frac{1}{a} - \frac{1}{\sqrt{L^2 + a^2}} \right) \hat{i} - \frac{kQ}{a \sqrt{L^2 + a^2}} \hat{j}$$

Note: This is harder than what you should expect to find on the exam. It is an example of the general case, however, where there is no cancellation of any components of $\vec{E}_p$ due to symmetry.
15. Point charges $+q_1$ and $-q_2$ are placed on diagonally opposite corners of a rectangle of sides $d_1$ and $d_2$. Point $A$ is the unoccupied corner a distance $d_1$ from the positive charge, and point $B$ is the other unoccupied corner.

(a) Determine the potential difference $V_B - V_A$.

(b) What is the electric potential energy of this system of two charges? Find the ratio of $q_1/q_2$ such that if a third charge $q_3$ is placed at $A$, the potential energy of the system would not change.

(c) How much work is done in moving a test charge $q_0$ from $A$ to $B$?

\[ V_B = V_{\text{at } B \text{ due to } q_1} + V_{\text{at } B \text{ due to } -q_2} \]

\[ V_A = V_{\text{at } A \text{ due to } q_1} + V_{\text{at } A \text{ due to } -q_2} \]

\[ \dot{W} = \frac{kq_1}{d_1} - \frac{kq_2}{d_2} - \frac{kq_1}{d_1} + \frac{kq_2}{d_2} \]

\[ = k (q_1 + q_2) \left( \frac{1}{d_2} - \frac{1}{d_1} \right) \]

(b) \[
U(q_1,-q_2) = -q_1 \cdot V_{\text{due to } q_1} \text{ at } (-q_2) = q_1 \cdot V_{\text{due to } -q_2} \text{ at } q_1
\]

\[
= \frac{kq_1q_2}{d_1^2 + d_2^2}
\]

\[
U(q_1,-q_2, q_3) = U(q_1,-q_2) + W_{\text{on } q_3 \text{ from } oo \text{ to } A}
\]

\[
W_{\text{on } q_3 \text{ from } oo \text{ to } A} = \text{ work done to bring } q_3 \text{ from } oo \text{ to } A = q_3 (V_A - V_B) = q_3 (V_{\text{at } A \text{ due to } q_1} \text{ and } (-q_2))
\]

\[
= q_3 \left( \frac{kq_1}{d_1} - \frac{kq_2}{d_2} \right) = 0
\]

Setting $W=0$ (\Rightarrow $U(q_1,-q_2, q_3) = U(q_1,-q_2) \Rightarrow q_1 = \frac{d_1}{d_2} q_2$).

(c) Work done in moving $q_0$ from $A$ to $B$ is

\[
W_{A\rightarrow B} = q_0 (V_B - V_A)
\]

\[
V_B = \frac{V_{\text{due to } q_1}}{at \ B} + \frac{V_{\text{due to } -q_2}}{at \ B} = \frac{kq_1}{d_2} - \frac{kq_2}{d_1}
\]

\[
V_A = \frac{V_{\text{due to } q_1}}{at \ A} + \frac{V_{\text{due to } -q_2}}{at \ A} = \frac{kq_1}{d_1} - \frac{kq_2}{d_2}
\]

\[
W_{A\rightarrow B} = q_0 \left[ \left( \frac{kq_1}{d_2} - \frac{kq_2}{d_1} \right) - \left( \frac{kq_1}{d_1} - \frac{kq_2}{d_2} \right) \right]
\]

\[
= kq_0 \left( \frac{1}{d_2} - \frac{1}{d_1} \right) (q_1 - q_2) 
\]
16. To probe the unknown charge of a nucleus, an alpha-particle of charge \( q = +2e \), where \( e \) is the fundamental charge, and mass, \( m_\alpha \), is launched directly toward it from far away with speed, \( v \). The alpha-particle reaches a point of closest approach before turning around. Assume the unknown nucleus remains stationary in this process. If the measured distance of closest approach of the alpha particle to the nucleus is \( d \), determine the charge of nucleus, \( Q = +Ze \), where \( Z \) is an integer. Express your answer in terms the electrostatic constant, \( k \), fundamental charge, \( e \), and the given quantities, \( m_\alpha \) and \( v \).

When the \( \alpha \)-particle is far from the nucleus:

\[
Q = +2e \quad \text{nucleus at rest} \quad q_\alpha = +2e
\]

\[
E_{\text{tot}} = K + U = \left( \frac{1}{2} m_\alpha v^2 + 0 \right) + \frac{k Q q_\alpha}{r}
\]

\[
= \frac{1}{2} m_\alpha v^2 + \frac{k(2e)(2e)}{r}
\]

\[
= \frac{1}{2} m_\alpha v^2
\]

At the distance of closest approach, \( \alpha \)-particle come to a stop before turning around and heading in the opposite direction:

\[
Q = +2e \quad q_\alpha = +2e \quad \text{both nucleus and } \alpha \text{-particle at rest}
\]

\[
E_{\text{tot}} = K + U = \left( 0 + 0 \right) + \frac{k Q q_\alpha}{r} = \frac{2kZe^2}{d}
\]

Since energy is conserved:

\[
\frac{1}{2} m_\alpha v^2 = \frac{2kZe^2}{d} \quad \Rightarrow \quad Z = \frac{1}{4} \frac{m_\alpha v^2 d}{ke^2}
\]