P521 Exam 1

1. **(6 points)** The following true or false questions concern orbital motion in a central force potential, $V(r)$. You must justify your answer.

   - (a) All orbits in the potential $V(r) = \frac{1}{2}kr^2$ are ellipses (or circles).
   - (b) Kepler’s second law ("equal areas in equal times") is true for all central potentials, not just the Kepler potential.
   - (c) A particle moves in an attractive $1/r$ potential. For a fixed total energy $E < 0$, the orbit with the maximum angular momentum is a circle.

2. **(9 points)** A point mass slides without friction inside the surface of revolution given by $z = A \cosh \rho/a$ in cylindrical polar coordinates, in the presence of a uniform gravitational field, $\vec{g} = -g\hat{z}$. $A$ and $a$ are constants.

   - (a) Construct the Lagrangian in terms of independent generalized coordinates. What is the cyclical variable and associated conserved generalized momentum? Obtain the equations of motion.

   - (b) Find the condition for a circular orbit, and determine the frequency of small oscillations about it. You may express your result in terms the radius of the circular orbit, $\rho_0$. *(Hint: Use the equations of motion obtained in (a). Recall that the Taylor series expansion of $f(\rho)$ is given by:)*

     $$f(\rho + \delta \rho) = f(\rho) + f'(\rho)\delta \rho + O(\delta \rho^2).$$

   - (c) Using the method of Lagrange multipliers, determine the normal force on the mass. What is the normal force when $m$ is in a circular orbit, $\rho = \rho_0$?

3. **(5 points)** The binary system, Cygnus X-1, consists of two stellar objects orbiting about their common center-of-mass under the influence of their mutual gravitational forces. One of the objects is a blue supergiant star, with mass $M_1 = 25M_\odot$ where $M_\odot$ is the mass of the Sun, while the other is believed to be a black hole, with mass $M_2 = 10M_\odot$. For what follows, assume that the distance of each object from their common center-of-mass remains fixed and is given by $R_1$ for the blue supergiant and $R_2$ for the black hole candidate. The period of this motion is $T$. You may ignore relativistic effects.

   - (a) Make a sketch of this system. Determine the length of the semi-major axis, $a$, of the Kepler orbit for the relative motion of the masses in terms of $R_1$ and $R_2$.

   - (b) The period of the Kepler orbit for the relative motion of the masses is given by $\tau = 2\pi a^{3/2}/\sqrt{G(M_1 + M_2)}$. Find $R_1$ and $R_2$ in terms of $T$, $G$, and $M_\odot$. 

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