Indiana University
Physics P521 Problem Set 1

1. Consider the functional given by the definite integral

\[ I[y(x)] = \int_a^b F(y, y', y'', x) \, dx, \]

which contains the first and second derivatives of \( y(x) \). To find the conditions for a stationary value, \( \delta I = 0 \), construct the variation of the integral and apply integration by parts twice. The imposed boundary conditions are \( y(x_a) = y_a \) and \( y(x_b) = y_b \). What is the resulting differential equation, and what additional boundary conditions must be satisfied for \( y(x) \) to be an extremal solution?

2. Find the extremals of the functional

\[ I[y(x), z(x)] = \int_0^{\pi/2} \left( y'^2 + z'^2 + 2yz \right) \, dx, \]

subject to \( y(0) = z(0) = 0 \) and \( y(\pi/2) = z(\pi/2) = 1 \). Using (for example) Mathematica, plot the values of the functional as a function of \( \epsilon \) taking

\[ y(x) = y^*(x) + \delta y(x) = y^*(x) + \epsilon \phi(x), \]
\[ z(x) = z^*(x) + \delta z(x) = z^*(x) + \epsilon \psi(x), \]

where \( y^*(x) \) and \( z^*(x) \) are the extremal solutions, and \( \phi(x) \) and \( \psi(x) \) are (continuous and differentiable) functions of your choice, satisfying appropriate conditions at the endpoints.

3. A cycloid is the path generated by a point \( P \) on the rim of circle that rolls without slipping along a horizontal line:

The parametric equations of the cycloid are \( x = R(\theta - \sin \theta) \), \( y = R(1 - \cos \theta) \).
A particle starts at rest at $A$ and slides down the cycloid $ABC$ in a uniform gravitational field to point $C$. The point $A$ is at $\theta = 0$, and the point $C$ is at $\theta = \theta_0$.

(a) Calculate the time it takes for the particle to reach $C$.
(b) Calculate the time it takes for the particle to slide along the straight line $ADC$, and show that it is greater than that for the cycloid.
(c) Now suppose $C$ is at $\theta = \pi$. Show that the time it takes for the particle released from rest at $A$ to slide down to $C$ is the same as that for one released from any intermediate point such as $B$ (at $\theta = \theta_i$). [Hint: Show that $t_{ABC}^2 - t_{ADC}^2 < 0$.]

The identities:

\[
1 - \cos \alpha = 2 \sin^2 \frac{\alpha}{2},
\]

\[
1 + \cos \alpha = 2 \cos^2 \frac{\alpha}{2},
\]

will be helpful in doing (b) and (c).

4. A mass $m$ is placed on a frictionless hemisphere of mass $M$ and radius $R$. $m$ starts at $\theta = 0$ at rest, and slides down the hemisphere.

(a) Assuming that the hemisphere is stationary, find the angle $\theta_\infty$, measured with respect to the vertical axis of the hemisphere, at which $m$ flies off.
(b) Now suppose the hemisphere is allowed to slide on the frictionless table. Find the angle $\theta_M$ at which it flies off. Prove that $\theta_M \leq \theta_\infty$, and that $\theta_M \to \theta_\infty$ as $M \to \infty$.

[Hints: Take the position $(x_m, y_m)$ of $m$ and the position $X_M$ of $M$ to be measured with respect to fixed lab coordinates, and use conservation of energy and...
momentum. What are the constraint equations relating $x_m$, $y_m$, $\theta$, $R$ and the position of the hemisphere, $X_M$, for $m$ to remain in contact with $M$? When $m$ loses contact with the hemisphere, what are horizontal and vertical components of its acceleration, $a_x$ and $a_y$, equal to?

5. A pendulum consisting of a rigid massless rod of length $\ell$ with a mass $m$ at one end moves in a uniform gravitational field $g$. The point of support is not stationary but moves in the vertical direction with a displacement $y(t)$ which is a given function of time.

(a) Find the Lagrangian and the equations of motion.

(b) Show that the equation is the same as that for pendulum with fixed support, with $g$ replaced by $g_{\text{eff}}(t)$. What is $g_{\text{eff}}(t)$?

[Hint: Denote the Cartesian coordinates of the mass $m$ by $(x_m, y_m)$, where $y_m(t) \neq y(t)$.]