Due Dec 10, 2004

Indiana University
Physics P521 Problem Set 12

1. A charged particle is constrained to move in a plane under the influence of a nonelectro-
   magnetic central potential, \( V = \frac{1}{2}kr^2 \), and a constant magnetic field, \( \vec{B} \), perpendicular
to the plane, so that the vector potential is given by \( \vec{A} = \frac{1}{2} \vec{B} \times r \).

(a) Find the Hamiltonian in polar coordinates.
(b) Set up the Hamilton-Jacobi equation for Hamilton's characteristic. Separate the
motion and reduce it to quadrature. Discuss the motion if the canonical momen-
tum \( p_r \) is zero at \( t = 0 \).

The Lagrangian for a charged particle in an electromagnetic field
is given by

\[
L = \frac{1}{2} m \dot{r}^2 - e \phi + \frac{e}{c} \dot{r} \cdot \vec{A}
\]

In the absence of an electric field, however, under the influence
of a nonelectromagnetic central potential, \( V(r) = \frac{1}{2}kr^2 \), with the
magnetic field given by a constant vector \( \vec{B} \) to the plane of the
motion:

\[
L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{e}{c} \dot{r} \cdot (\vec{B} \times \dot{r}) - \frac{1}{2} kr^2
\]

(where we have used the fact that with \( V=V(r) \) and \( \vec{B} \perp \vec{r} \)
the motion remains planar.)

First, find the Hamiltonian,

\[
\dot{r} \cdot (\vec{B} \times \dot{r}) = (\dot{r} \hat{r} + r \dot{\theta} \hat{\theta}) \cdot (\vec{B} \hat{r} \times \dot{r} \hat{r}) = (\dot{r} \hat{r} + r \dot{\theta} \hat{\theta}) \cdot B \hat{r} \hat{\theta} = B r^2 \dot{\theta}
\]

\[
\Rightarrow \quad L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{e}{c} \frac{r^2 \dot{\theta}}{2} - \frac{1}{2} kr^2
\]

\[
p_r = \frac{\partial L}{\partial \dot{r}} = m \dot{r} \Rightarrow \quad \dot{r} = p_r / m
\]

\[
p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} + \frac{e}{c} \frac{r^2 \dot{\theta}}{2} \Rightarrow \quad \dot{\theta} = \frac{1}{m r^2} \left( p_\theta - \frac{e}{c} \frac{r^2 \dot{\theta}}{2} \right)
\]

\[
\Rightarrow \quad H = -(L - \frac{1}{r} p_r \dot{r}^2)
\]

\[
= -\frac{1}{2m} \left[ \dot{r}^2 + \frac{4m}{e} (p_\theta - \frac{e}{c} \frac{r^2 \dot{\theta}}{2})^2 \right] - \frac{e}{2mc} \left( p_\theta - \frac{e}{c} \frac{r^2 \dot{\theta}}{2} \right) + \frac{1}{2} kr^2
\]

\[
+ \frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} \left( p_\theta - \frac{e}{c} \frac{r^2 \dot{\theta}}{2} \right)^2 + \frac{1}{2} kr^2
\]

The Hamilton-Jacobi equation for the generating function, \( S \) (Hamilton's
principal function), which generates a canonical transformation to
variables \( (Q_r, P_r) \) such that the new Hamiltonian \( K(Q_r, P_r) = 0 \) is:

\[
H(Q_r, P_r, p_r, p_\theta) + \frac{\partial S}{\partial t} = 0 \quad (= K)
\]

where \( \frac{\partial S}{\partial Q_r} = p_r \) and \( \frac{\partial S}{\partial P_r} = Q_r \;
\]

\[
\Rightarrow \quad H(r, \theta, \frac{\partial S}{\partial \theta}, \frac{\partial S}{\partial \theta}, \frac{\partial S}{\partial t}) + \frac{\partial S}{\partial t} = 0
\]

Writing \( \dot{S}(r, \theta, t) = W(r) + W_\theta(\theta) + W_t(t) \), i.e., assuming that
\( S \) is completely separable:

\[
H(r, \theta, W_r, W_\theta, W_t) + \frac{\partial S}{\partial t} = 0
\]
\[ L = \frac{1}{2} m (r^2 + r^2 \dot{\theta}^2) + \frac{eB}{2c} r^2 \dot{\theta} = \frac{1}{2} kr^2 \]

\[ p_r = \frac{2L}{\dot{r}} = \frac{m}{r} \Rightarrow \dot{r} = \frac{p_r}{m} \]

\[ p_{\theta} = \frac{2L}{\dot{\theta}} = \frac{m^2 \dot{\theta} + eBR}{2c} \Rightarrow \dot{\theta} = \frac{1}{m^2} (p_{\theta} - \frac{eBR}{2c}) \]

\[ H = -\left( L - \sum_{i=1}^{2} p_i \dot{q}_i \right) \]

\[ = \frac{1}{2m} \left[ p_r^2 + \frac{1}{r^2} \left( p_{\theta} - \frac{eBR}{2c} \right)^2 \right] - \frac{eB}{2mc} \left( p_{\theta} - \frac{eBR}{2c} \right) + \frac{1}{2} kr^2 \]

\[ + \frac{p_r^2}{m} + \frac{p_{\theta}}{mr^2} \left( p_{\theta} - \frac{eBR}{2c} \right) \]

\[ = \frac{1}{2m} \left[ p_r^2 + \frac{1}{r^2} \left( p_{\theta} - \frac{eBR}{2c} \right)^2 \right] + \frac{1}{2} kr^2 \]

The Hamilton-Jacobi equation for the generating function \( S \) (Hamilton's principal function), which generates a canonical transformation to variables \( \{ q_i, p_i \} \) such that the new Hamiltonian \( K(q, p) = 0 \) is:

\[ H(q, p) + \frac{2S}{\dot{t}} = 0 \quad (\approx K) \]

where \( \frac{2S}{\dot{t}} = p_i \) and \( \frac{2S}{\dot{q}_i} = q_i \).

\[ \Rightarrow H(r, \theta, \frac{2S}{\dot{r}}, \frac{2S}{\dot{\theta}}) + \frac{2S}{\dot{t}} = 0 \]

Writing \( S(r, \theta, t) = W_r(r) + W_\theta(\theta) + W_t(t) \), it is assumed that \( S \) is completely separable:

\[ \Rightarrow \frac{dW_r}{dr} + \frac{1}{r^2} \left( \frac{dW_\theta}{d\theta} - \frac{eB r}{2c} \right)^2 + \frac{1}{2} kr^2 + \frac{dW_t}{dt} = 0 \]

Setting \( \frac{dW_r}{dr} = -\alpha_3, \frac{dW_\theta}{d\theta} = \alpha_2 \Rightarrow W_r = -\alpha_3 t \quad (1) \]

\[ \frac{dW_\theta}{d\theta} = \alpha_2 \Rightarrow W_\theta = \alpha_2 \theta + \theta_0 \quad (2') \]

\[ \begin{align*}
\frac{dW_t}{dt} &= \alpha_3 \frac{dW_r}{dr} + \alpha_2 \frac{dW_\theta}{d\theta} \\
&= \alpha_3 (-\alpha_3 t) + \alpha_2 (\alpha_2 \theta + \theta_0) \\
&= \alpha_3^2 t - \alpha_3 \alpha_2 \theta - \alpha_2 \theta_0
\end{align*} \]

\[ \Rightarrow \frac{dW_t}{dt} = \alpha_3^2 t - \alpha_2 \frac{eBR}{2c} t - \alpha_2 \theta_0 = 2m \alpha_3 \]

With (1) - (3), the solution to the HJ equations is reduced to quadratures.

From (3):

\[ W_\theta = \pm \int dr' \left\{ 2m \left( \alpha_3^2 - \frac{1}{2} kr^2 \right) - \frac{1}{r^2} \left( \alpha_2 - \frac{eBR}{2c} \right)^2 \right\}^{1/2} \]

Now, from (2'), \( \frac{dW_\theta}{d\theta} = \alpha_2 \), if \( p_{\theta}(t=0) = 0 \), then \( \alpha_2 = 0 \) and \( p_{\theta} = 0 \) for all time (\( \theta \) is a cyclic variable, so \( p_\theta \) is conserved.)

Solving for \( r(t) \) with \( \alpha_2 = 0 \):

\[ \dot{p}_3 = \frac{2S}{\dot{r}} = -t + 2W_t = -t \pm \int dr' \frac{m}{\left\{ 2m \alpha_3 - \frac{eBR}{2c} \right\}^{1/2}} \]

Let \( \alpha = \left[ \frac{mk + (eB)^2}{2c} \right] / 2m \alpha_3 \).
2. Construct the Hamiltonian for the one-dimensional harmonic oscillator of mass \( m \) and spring constant \( k \). Determine the value of the constant \( C \) such that the following equations define a canonical transformation from the old variables \((q, p)\) to the new variables \((Q, P)\):

\[
Q = C(p + i\omega q), \\
P = C(p - i\omega q),
\]

where \( \omega = (k/m)^{1/2} \). What is the generating function, \( F(q, P) \), for this transformation? Find Hamilton's equations of motion for the new variables and integrate them. Hence find the solution to the original problem.

\( (R, P) \) must satisfy the fundamental Poisson brackets. Recall that Poisson brackets are canonical invariants, i.e., \( \{A, B\}_P \neq \{A, B\}_Q \). Hence, require

\[
\{Q, P\}_P = \frac{\partial Q}{\partial q} \frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p} \frac{\partial P}{\partial q} = 0
\]

\[
\{Q, P\}_P = \begin{pmatrix} 2Q & 2P \\ 2Q & -2P \end{pmatrix} \approx \begin{pmatrix} 2i\omega & 0 \\ 0 & 2i\omega \end{pmatrix}
\]

\( iC^2 \omega \left[ 1 + 1 \right] = 2i\omega C^1 = 1 \Rightarrow C = (2i\omega)^{-1/2} \)

With a hyperbolic generating function, \( F = F_i(q_i, p_i, t) - q_i p_i \), we have

\[
p_c = \frac{\partial F_i}{\partial q_i}, \quad q_c = \frac{\partial F_i}{\partial p_i} \quad \text{and} \quad H = H + \frac{\partial F_i}{\partial t}
\]

Using \( p = \frac{1}{C} \hat{P} + i\omega q = p(q, P) \), we find

\[
\frac{1}{C} \hat{P} + i\omega q = \frac{2F_i}{\partial q} \Rightarrow F_i(q, P) = \frac{1}{C} P^2 + \frac{1}{2} i\omega q^2 + \hat{g}(P)
\]

Now, \( \frac{2F_i}{\partial P} = \frac{1}{C} q + \hat{g}'(P) = Q = C_p + i\omega C_q = P + 2i\omega C_q \Rightarrow \hat{g}'(P) = P \Rightarrow \hat{g}(P) = \frac{1}{2} P^2 \)

Hence, \( F_i(q, P) = \sqrt{2i\omega} P q + \frac{1}{2} P^2 + \frac{1}{2} i\omega q^2 \).
Note that if you had started with \( Q = 2 \tilde{F} \), \( C(p + i m q) \), you would before integrating over \( p \) to find \( \tilde{F}(q, \tilde{L}) \), you would have had to first express \( p \) in terms of \( q \) and \( \tilde{L} \).

The new Hamiltonian is given by

\[
\tilde{H} = H(p, \tilde{L}, q, \tilde{L}, q, t) + \frac{\partial F}{\partial \tilde{L}}
\]

Now, \( H(p, q) \) for the 1D harmonic oscillator is:

\[
L = T - V = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} k q^2 = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2
\]

\[
H = p^2 \frac{1}{2m} - \left( \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2 \right) = \frac{1}{2} m \dot{q}^2 + \frac{1}{2} m \omega^2 q^2
\]

Finding \( (p, \tilde{L}, q, \tilde{L}, q, t) \):

\[
p = \frac{\sqrt{2}}{2} \left( \tilde{L} + q \right)
\]

\[
q = \frac{\sqrt{2}}{2m} \left( \tilde{L} - q \right)
\]

\[
\tilde{H} = \frac{1}{8m c} \left( (p + \tilde{L})^2 - (p - \tilde{L})^2 \right) = \frac{1}{2m c^2} \tilde{L} q = i \omega \tilde{L} q
\]

Hamilton's equations of motion become:

\[
\dot{Q} = \frac{\tilde{H}}{2p} = i \omega \tilde{L} \quad \Rightarrow \quad Q(t) = Q(0) e^{i \omega t}
\]

\[
\dot{\tilde{L}} = -i \omega q \quad \Rightarrow \quad \tilde{L}(t) = \tilde{L}(0) e^{-i \omega t}
\]

\[
Q_0, \tilde{L}_0 \) are determined from initial conditions: \((q(0) = q_0, p(0) = p_0)\)

\[
\begin{align*}
Q_0 &= C(p_0 + i m q_0) \\
\tilde{L}_0 &= C(p_0 - i m q_0)
\end{align*}
\]

Therefore, \((q(t), p(t))\) become:

\[
p(t) = \frac{1}{2} \left\{ (p_0 - im q_0) e^{-i \omega t} + (p_0 + im q_0) e^{i \omega t} \right\}
\]

\[
= \frac{1}{2} \left\{ 2 p_0 \cos \omega t + im q_0 (2i \sin \omega t) \right\}
\]

\[
= p_0 \cos \omega t - m q_0 \sin \omega t
\]

\[
q(t) = \frac{1}{2i m \omega \sin \omega t} \left\{ (p_0 - im q_0) e^{-i \omega t} - (p_0 + im q_0) e^{i \omega t} \right\}
\]

\[
= \frac{1}{2i m \omega} \left\{ 2 i p_0 \sin \omega t - 2 i m q_0 \cos \omega t \right\}
\]

\[
= p_0 \sin \omega t - m q_0 \cos \omega t,
\]
3. Find the values of \( \alpha \) and \( \beta \) for which the transformation

\[
Q = q^{\alpha} \cos(\beta p), \quad P = q^{\alpha} \sin(\beta p),
\]

is canonical. For these values of \( \alpha \) and \( \beta \), construct a generating function \( F_3(p, Q) \) for the canonical transformation.

Require \( (Q, P) \) to satisfy the fundamental Poisson bracket:

\[
\{Q, P\} = \mathbb{I}, \quad \{Q, Q\} = \{P, P\} = 0
\]

\[
\{Q, P\}_{\alpha \beta} = \frac{\partial Q}{\partial \alpha} \frac{\partial P}{\partial \beta} - \frac{\partial Q}{\partial \beta} \frac{\partial P}{\partial \alpha} = \frac{\alpha}{\beta} \left( q^{2\alpha-1} \cos(\beta p) + q^{2\alpha-1} \sin(\beta p) \right) = \alpha \beta q^{2\alpha-1} = 1.
\]

\( 2\alpha - 1 = 0 \Rightarrow \alpha = \frac{1}{2} \)

\( \alpha \beta = 1 \Rightarrow \beta = 2 \).

For a type-3 generating function, \( F = F_3(p, Q, t) + q_i \dot{q}_i \):

\[
\frac{dF}{dt} = (\frac{\partial F}{\partial \dot{q}_i} + q_i) \dot{q}_i + \frac{\partial F}{\partial q_i} \dot{Q}_i + P_i \ddot{q}_i + \ddot{P}_i.
\]

For \( F \) take the generator of a canonical transformation:

\[
\sum_i p_i \dot{q}_i - H = \sum_i P_i \dot{Q}_i - \tilde{H} + \frac{dE}{dt}
\]

\( H = \sum_i (P_i + \beta q_i) \dot{Q}_i - \tilde{H} + (\frac{\partial F}{\partial \dot{q}_i} + q_i) \dot{q}_i + \frac{\partial F}{\partial q_i} \dot{Q}_i + \frac{\partial F}{\partial \dot{Q}_i} \dot{P}_i
\]

\[
P_i = -\beta q_i, \quad q_i = -\frac{\partial F}{\partial \dot{Q}_i} \quad \text{and} \quad \tilde{H} = H + \frac{\partial F}{\partial \dot{Q}_i}
\]

Now, \( Q = q^{\frac{1}{2}} \cos(2p) \)

\[
\begin{align*}
L &= q^{\frac{1}{2}} \sin(2p) \\
\frac{L}{Q} &= \tan(2p) \Rightarrow L = Q \tan(2p)
\end{align*}
\]

Using, \( L = Q \tan(2p) = \frac{2F}{Q} \Rightarrow F_3(p, Q) = -\frac{1}{2} Q^2 \tan(2p) + q(p) \)

Therefore,

\[
q = -\frac{\partial F_3}{\partial p} = -Q^2 \frac{1}{\cos^2(p)} + \frac{q(p)}{p} = -\frac{Q^2}{\cos^2(p)}
\]

\( q(p) = 0 \Rightarrow q(p) = \text{constant} = 0 \)

So, \( F_3(q, p) = -\frac{1}{2} Q^2 \tan(2p) \).
4. Show from the Poisson bracket condition for conserved quantities that the Laplace-
Runge-Lens vector $\vec{A}$

$$\vec{A} = \vec{p} \times \vec{L} - \frac{mkr}{r}$$

is a constant of motion for a mass $m$ moving under the Kepler potential. Also, show
this by taking the time derivative of this quantity.

(a) In terms of Poisson brackets, the time rate of change of a
dynamical variable, in this case the Laplace-Runge-Lens vector
$\vec{A}$, is given by

$$\frac{d\vec{A}}{dt} = [\vec{A}, H] + \frac{\partial \vec{A}}{\partial t}$$

Since $\frac{d\vec{A}}{dt} = 0$, to show that $\vec{A}$ is conserved, we need
$$\frac{\partial \vec{A}}{\partial t} = 0.$$

First, the Hamiltonian $H$ is given by:

$$H = (L - \sum_i p_i \dot{q}_i)$$

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2)$$

[Note that for a central potential, $V = V(r)$, angular momentum is
conserved, and the motion takes place
in a plane $\theta$ to the constant angular
momentum vector.]}

The canonical coordinates and momenta are:

$$\vec{q} = (q_1, q_2) = (r, \theta)$$

$$\vec{p} = (p_1, p_2) = \left( \frac{\partial L}{\partial \dot{r}}, \frac{\partial L}{\partial \dot{\theta}} \right) = \left( \dot{r}, m^2 \dot{\theta} \right)$$

Note that the canonical momenta $p_r, p_\theta$ are NOT the
 corresponding components of the linear momentum $\vec{p}$,

$$\Rightarrow \quad H = \frac{1}{2m} \left( \frac{p_r^2}{r^2} + \frac{p_\theta^2}{r^2} \right) - \frac{k}{r} (\dot{r} + V)$$

[For $V = V(1/r)\dot{r}$, i.e., potential independent of $q_i$, and
with time-independent constants, we have
$$H = T + V = energy.$$]

Next, write $\vec{A}$ in polar coordinates:

$$\vec{A} = \vec{p} \times \vec{L} - mk \hat{r}$$

$$= m (\dot{r} \hat{r} + r \dot{\theta} \hat{\theta}) \times \dot{r}^2 \hat{r} + m \dot{\theta} \hat{\theta} - mk \hat{r}$$

$$= m \dot{r} \hat{r} \dot{r} \hat{\theta} + m \dot{\theta} \hat{\theta} \dot{r} \hat{r} - mk \hat{r}$$

$$= m \left( (m^2 \dot{r}^2 - k) \hat{r} - m \dot{r} \dot{\theta} \hat{\theta} \right) = A_r \hat{r} + A_\theta \hat{\theta}$$

The Poisson bracket $[\vec{A}, H]$ is given by:

$$[\vec{A}, H] = \sum_i \left( \frac{\partial A_i}{\partial q_i} \frac{\partial H}{\partial \dot{q}_i} - \frac{\partial A_i}{\partial \dot{q}_i} \frac{\partial H}{\partial q_i} \right)$$

Writing $(A_r, A_\theta)$ as a function of the canonical coordinates
and momenta:

$$A_r = \frac{p_r^2}{r} - mk ; \quad A_\theta = -p_r p_\theta$$
and recalling
\[ \hat{r} = \cos \theta \hat{x} + \sin \theta \hat{y} \]
\[ \hat{\theta} = -\sin \theta \hat{x} + \cos \theta \hat{y} \]
we have
\[ 2\hat{r} = \hat{\theta} \quad ; \quad 2\hat{\theta} = -\hat{r} \]
Hence,
\[ [\hat{A}, \hat{H}] = \frac{2}{\theta \phi} \frac{\partial (A_r \hat{r} + A_\theta \hat{\theta})}{\partial A_r} \frac{2H}{\partial \theta} \]
\[ + \frac{2}{\theta} \frac{\partial (A_r \hat{r} + A_\theta \hat{\theta})}{\partial A_\theta} \frac{2H}{\partial \theta} \]
\[ \hat{\theta} : A_r \frac{\partial H}{\partial A_r} - 2A_\theta \frac{\partial H}{\partial A_\theta} = \left( \frac{\rho^2}{r} - \frac{m}{r^2} \right) \left( \frac{\rho_0}{m r^2} \right) + \rho_0 \left( \frac{\rho^2}{m r^2} + \frac{k}{r^2} \right) \]
\[ = 0 \]
\[ \hat{r} : -A_\theta \frac{\partial H}{\partial A_r} + 2A_r \frac{\partial H}{\partial A_\theta} = \frac{\rho_0}{m} \rho_0 \left( \frac{\rho_0}{m r^2} \right) - \frac{\rho^2}{r} \frac{\rho_0}{m} \]
\[ = 0 \]
Hence, \[ [\hat{A}, \hat{H}] = 0 \]

b) We can also show that \( \hat{A} \) is conserved by direct differentiation (and using the equations of motion):
\[ \hat{r} = \frac{2H}{\theta \phi} \frac{\rho}{m} \]
\[ \hat{\theta} = -\hat{r} \quad ; \quad \hat{\theta} \hat{r} = -\frac{\rho^2}{m r^2} + \frac{k}{r^2} \]
\[ \hat{r} = \frac{2H}{\theta \phi} \frac{\rho}{m} \]
\[ \hat{\theta} = \frac{2H}{\theta \phi} \frac{\rho}{m r^2} \quad ; \quad \frac{\rho_0}{m r^2} = 0 \]