1. We have a pendulum (mass \(m\) at the end of the massless rod of length \(l\)). The top support of the pendulum is attached to two springs of spring constant \(k\) (which are fixed at the opposite ends), and slides back and forth along the \(x\) axis. Gravity acts downward.

![Diagram of a pendulum with springs and mass](image)

(a) Assuming the top support of the pendulum is massless, find the constraint equations.

(b) For small oscillations of the system, write down the Lagrangian using the Lagrange method of undetermined multipliers, retaining terms up to and including second order in the small quantities. (Assume all other physical parameters, such as \(k/m\) and \(g/l\) are of order unity.)

(c) Obtain the corresponding equations of motion, and show that the system acts like a simple pendulum with an effective length, \(l_{eff}\), and find that effective length.

2. A bead of mass \(m\) slides without friction under gravity along a wire which has the shape \(y(x) = ax^2 - bx^3\).

(a) Write down the Lagrangian in terms of \(x\) and \(y\). Then, using the constraint equation, obtain a Lagrangian in only one independent (generalized) coordinate. Find the equation of motion for that coordinate.

First, let the support be massive, with mass \(M\). Then, the constraint equation is that the length of the string is fixed:

\[
(x_m - x_s)^2 + y_m^2 = l^2, \quad x_m - x_s = \sqrt{l^2 - y_m^2} = 0
\]

Using Lagrange multipliers to account for the constraint, we will write \(\overline{\mathcal{L}}\) in terms of the dependent coordinate:

\[
\overline{\mathcal{L}} = \left( T_x + T_m \right) - (V_x + V_m) = \frac{1}{2} m x_s^2 + \frac{1}{2} m \left( \dot{x}_m^2 + \dot{y}_m^2 \right) - 2 \left( \dot{x}_s k x_s \right) - (-mg y_m) + \lambda (x_m - x_s - \sqrt{l^2 - y_m^2})
\]

\(\lambda\) is the constraint.

Therefore, \(\overline{\mathcal{L}} = \overline{\mathcal{L}}(x_m, y_m, x_s, \lambda)\). The Euler-Lagrange equations are:

\[
x_s : \quad \dot{y}_s = 2 k x_s + 2 = 0 \quad (\lambda)
\]

\[
x_m : \quad m \ddot{x}_m - \lambda = 0 \quad (2)
\]

\[
y_m : \quad m \ddot{y}_m - mg - \lambda \frac{\dot{y}_m^2}{\sqrt{l^2 - y_m^2}} = 0 \quad (3)
\]

\[
\lambda : \quad x_m - x_s = \sqrt{l^2 - y_m^2} \quad (4) \Rightarrow \begin{cases} x_m - x_s = \lambda \sin \theta \\ y_m = \lambda \cos \theta \\ \dot{x}_s = x_s + \lambda \dot{\theta} \cos \theta \\ \dot{y}_s = -\lambda \dot{\theta} \sin \theta \end{cases}
\]

Therefore, we have used \((4)\) to reduce \((x_m, y_m, x_s) \rightarrow (\theta, x_s)\).
\( (3) \) \quad m \dddot{x} + m \ddot{\theta} \cos \theta - m \dot{\theta}^2 \sin \theta - \lambda = 0 \quad (\alpha') \\
\( (3') \) \quad -m \dddot{\theta} \sin \theta - m \dot{\theta}^2 \cos \theta - mg - \lambda \cos \theta = 0 \quad (3'') \\
(1'), (2'), and (3') constitute 3 equations for 3 unknowns \( (x, \theta, \lambda). \)

If the displacement of the pendulum from the vertical is small,

\[
\cos \theta = 1 - \frac{1}{2} \theta^2 + o(\theta^4) \\
\sin \theta = \theta + o(\theta^3)
\]

then, \( m \dddot{x} + 2k \dot{x} + \lambda = 0 \) \quad (1) \n\( m \dddot{\theta} + m \dot{\theta} \ddot{\theta} - \lambda \ddot{\theta} = 0 \) \quad (2'') \n\( m \dddot{\theta} - m \dot{\theta} \ddot{\theta} - m \dot{\theta}^2 \cos \theta - mg = 0 \) \quad (3'')

\( (1'), (2'') \) \quad \Rightarrow \quad (m + m_s) \dddot{x} - 2k x + m \dot{\theta} = 0 \\
\( (1), (2') \) \quad \Rightarrow \quad -\lambda (m + m_s) = n_s (2k x - m \dot{\theta}) \\
\lambda = \frac{n_s (2k x - m \dot{\theta})}{m + m_s}

\( m \dddot{x} + k \ddot{x} + m (n_s \dot{\theta} - 2k x) = 0 \)
\( m \dddot{\theta} + m \dot{\theta} \ddot{\theta} + 2k (m + m_s) \dot{x} + m \dot{\theta}^2 - 2m_k x = 0 \)
\( m \dddot{x} + 2k m \dot{x} + m \dot{\theta} = 0 \)
\( m \dddot{\theta} + 2k \dot{\theta} + m \dot{\theta} = 0 \)

\( (3') \) \quad \Rightarrow \quad (m + m_s) \dddot{x} + 2k \dot{x} + m \dot{\theta} = 0 \quad (5) \\
(3'') \quad \Rightarrow \quad (m + m_s) \dddot{\theta} + m \dot{\theta} \ddot{\theta} - 2m_k x = 0 \quad (6) \)

(5) and (6) constitute coupled equations for \( x(t) \) and \( \theta(t) \), valid for small \( \theta \).

\( (6) \) \quad \Rightarrow \quad x = (m + m_s) \frac{\dot{\theta}}{2k} + \frac{m \dot{\theta}}{2k} \quad \Rightarrow \quad \dot{\theta} = \frac{d^2\theta}{dt^2} \frac{m \dot{\theta}}{2k} \quad \Rightarrow \quad \theta = \frac{m \dot{\theta}}{2k} \quad \Rightarrow \quad \theta = \frac{m \dot{\theta}}{2k} \\
\dddot{\theta} = \frac{(m + m_s) \dddot{\theta}}{2k} + \frac{m \dot{\theta} \ddot{\theta}}{2k} + \frac{2k (m + m_s) \dot{x}}{2k} + \frac{2k m \dot{\theta}}{2k} \quad \Rightarrow \quad \dot{\theta} = \frac{d^2\theta}{dt^2} \frac{m \dot{\theta}}{2k}

Let \( \theta(t) = A \cos (\omega t + \phi) \):
\( \dddot{\theta} = -\omega^2 \theta \quad \Rightarrow \quad \theta^{(v)} = g \theta \)
\(-\frac{(m + m_s) \dddot{\theta}}{2k} + \frac{m \dot{\theta} \ddot{\theta}}{2k} + \frac{2k (m + m_s) \dot{x}}{2k} + \frac{2k m \dot{\theta}}{2k} = 0

Let \( \omega^2 = g \); the general solution is given by:
\[ a x^2 + b y + c = 0 \]
\( a = \frac{(m + m_s) m \dot{\theta}}{2k} \)
\( b = -(m + m_s) \left(1 + \frac{(m + m_s) \dddot{\theta}}{2k}\right) \)
\( c = \frac{(m + m_s) \dot{x}}{2k} \)
For $m_2 = 0$:

$$\omega^2 = \frac{-c(m_2 = 0)}{b(m_2 = 0)} = \frac{1}{l^2} = \frac{1}{l} \frac{1}{\text{left}}$$

where $l_{\text{left}} = l \left(1 + \frac{m_2}{2k}\right)$. As $k \to \infty$, i.e., in the limit of stationary support, $l_{\text{left}} \to l$, as expected.

**Note:** $m_2 \to 0$ is NOT equivalent to $\dot{x}_2 = 0$; rather, from (1):

$$-2kx_3 + 2 = 0$$

which is equivalent to $(F_x)_{\text{support}} = 0$, since $2 = T_x$ is the horizontal component of the tension in the string.

Although we are not asked to find the tension $T_x$, it can be obtained from (2') and (3'):

$$T = T_x \sin \theta = \sin \theta \left[ m \ddot{x}_3 \sin \theta - ml \dot{\theta}^2 - mg \cos \theta \right]$$

Note that $T$ with a moving support is NOT the same as that with a stationary support ($\dot{x}_2 = 0$). However, in both cases, for small oscillations:

$$T = -mg + O(\theta^3).$$

Alternatively, we can use Lagrange multiplier to impose (1) with $m_2 \to 0$, and take into account constant string length by using the single generalized coordinate $\theta$ (instead of $(x_3, x_3')$).

In (1), $x$ is $x$-component of string tension $\dot{x} = T_x$. For the simple pendulum with stationary support:

$$L = \frac{1}{2} m \left( \dot{x}_2^2 + \dot{y}_2^2 \right) + mg \dot{y}_2 + \frac{1}{2} \left( \dot{x}_2^2 + \dot{y}_2^2 - l^2 \right).$$

**Cartesian**

$$L = \frac{1}{2} m \left( \dot{r}^2 + r^2 \dot{\theta}^2 \right) + m g r \cos \theta + \frac{1}{2} \left( r^2 - l^2 \right).$$

**Polar**

$$r: \frac{d}{dt} \left( \frac{2F}{2r} \right) - \frac{2F}{2r} = 0 \Rightarrow m \ddot{r} - m r \ddot{\theta} - mg \cos \theta - F = 0.$$

$$\theta: \frac{d}{dt} \left( \frac{2F}{2\theta} \right) - \frac{2F}{2\theta} = 0 \Rightarrow m \ddot{\theta} + 2 m r \ddot{r} + m g \cos \theta = 0.$$
\[ L = \frac{1}{2} m \left( \dot{x}^2 + \dot{z}^2 + 2 l \dot{x} \dot{z} \cos \theta \right) - k x_s^2 + m g l \cos \theta + 2 l \left[ 2 k x_s - \left( m \ddot{\theta}^2 + m g \cos \theta \right) \sin \theta \right] \]

where we have used \( V = 2 \left( \frac{1}{2} k x_s^2 \right) - m g z \). For small oscillation angles, \( \theta \ll 1 \), retain terms up to and including second order. Note that for oscillation frequencies, \( \omega, \phi \), of order unity, terms such as \( \theta \dot{\theta}^2 \sim \omega^2 \theta_0^2 \sim O(\theta_0^3) \) can be neglected, where \( \theta_0 \) is the amplitude of oscillation.

\[ L = \frac{1}{2} m \left( \dot{x}^2 + \dot{z}^2 + 2 l \dot{x} \dot{z} \cos \theta \right) - k x_s^2 + m g l \cos \theta + 2 l \left[ 2 k x_s - \frac{m g \sin \theta \cos \theta}{2 m g \sin \theta} \right] \]

\[ x_s = \frac{1}{2} \left( \frac{\ddot{x}}{2 x_s} \right) - \frac{\ddot{z}}{2 x_s} = 0 \]

\[ m \ddot{x} + m l \ddot{\theta} \cos \theta - m l \dot{\theta}^2 \sin \theta + 2 k x_s - 2 k z = 0 \] (1)

\[ \dot{\theta} = \frac{1}{l} \left( \frac{\ddot{x}}{\ddot{\theta}} \right) - \frac{l}{2} \theta = 0 \]

\[ m l \ddot{\theta} + m l z \dot{\theta} \cos \theta - m l \dot{\theta} \dot{\theta} \sin \theta + m g l \cos \theta + \lambda m g l \cos \theta = 0 \] (2)

\[ 2 k x_s = \frac{1}{2} m g \sin \theta \] (3)

Applying small angle approximations:

(3) \( x_s \approx \frac{m g \theta}{2 k} \)

(1) \( \frac{m g \ddot{\theta}}{2 k} + \frac{l \ddot{\theta}}{\theta} + g \theta - \frac{2 k}{m} \lambda = 0 \)
2. A bead of mass \(m\) slides without friction under gravity along a wire which has the shape \(y(x) = ax^2 - bx^2\).

(a) Write down the Lagrangian in terms of \(x\) and \(y\). Then, using the constraint equation, obtain a Lagrangian in only one independent (generalized) coordinate.

Find the equation of motion for that coordinate.

(b) Now leave the Lagrangian in terms of the two variables \(x\) and \(y\). Find the equations of motion for \(x\) and \(y\) using Lagrange's method of undetermined multipliers.

Show that your answer in (a) is equivalent to these equations.

(c) Find the normal force of constraint. (Optional: For \(y(x)\) given instead by \(y(x) = \sqrt{R^2 - x^2}\), find the point at which the normal force vanishes, and compare with Problem 1(a)., FS #1.)

(2.9)

\[ L = \tfrac{1}{2}m \left( \dot{x}^2 + \dot{y}^2 \right) - mg \, y_m \]

\[ \text{Constraint:} \quad y_m = ax_m^2 - bx_m^2 \]

\[ \dot{y}_m = x_m \, x_m \, (3 \, ax_m^2 - 2b) \]

The Lagrangian becomes:

\[ L = \tfrac{1}{2}m \dot{x}_m^2 \left\{ 1 + x_m^2 \left( 3 \, ax_m^2 - 2b \right) \right\} - mg \left( ax_m^2 - bx_m^2 \right) \]

\[ \dot{x}_m^2 \left( 9a^2x_m^4 + 4b^2x_m - 12bx_m^2 \right) \]

Euler-Lagrange equation for independent generalized coordinate \(x_m\):

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_m} \right) - \frac{\partial L}{\partial x_m} = 0 \]

\[ \frac{\partial L}{\partial \dot{x}_m} = mx_m \left\{ 1 + x_m^2 \left( 3 \, ax_m^2 - 2b \right) \right\} = mx_m \left\{ 1 + x_m^2 \left( 9a^2x_m^4 + 4b^2x_m - 12bx_m^2 \right) \right\} \]

\[ \frac{\partial L}{\partial \dot{x}_m} = \frac{1}{2}m \dot{x}_m \left\{ 36a^2x_m^4 + 8b^2x_m - 36ax_m^2 \right\} - mg \left( 3ax_m^2 - 2bx_m \right) \]

\[ \Rightarrow \dot{y}_m \left\{ 1 + x_m^2 \left( 3ax_m^2 - 2b \right) \right\} = \dot{x}_m \left\{ 36a^2x_m^4 + 8b^2x_m - 36ax_m^2 \right\} \]

\[ - \dot{x}_m \left\{ 9a^2x_m^4 + 4b^2x_m - 12bx_m^2 \right\} + mg \left( 3ax_m^2 - 2bx_m \right) = 0 \]

(2.10)

\[ \ddot{x}_m \left[ 1 + x_m^2 \left( 3ax_m^2 - 2b \right) \right] + 2 \dot{x}_m \dot{x}_m \left( 3ax_m^2 - 2b \right) \]

\[ + \frac{1}{2}x_m \left( 3ax_m^2 - 2b \right) = 0 \]

(b) Using Lagrange multipliers:

\[ \bar{L} = \frac{1}{2}m \left( \dot{x}_m^2 + \dot{y}_m^2 \right) - mg \, y_m + \lambda \left( y_m - ax_m^2 + bx_m \right) \]

\[ x_m : \quad \dot{x}_m + 3ax_m^2 - 2bx_m = 0 \]

\[ \ddot{x}_m + \dot{y}_m = 0 \]

\[ \dot{y}_m = \dot{y}_m + mg - \lambda = 0 \]

\[ \lambda = \frac{1}{2}m \left( \dot{y}_m^2 + \dot{y}_m \right) \]

From (a):

\[ \dot{y}_m = (3ax_m^2 - 2bx_m) \dot{x}_m \]

\[ \ddot{y}_m = (3ax_m^2 - 2bx_m) \ddot{x}_m + (6ax_m - 2b) \dot{x}_m^2 \]

From (b):

\[ \lambda = m \left( \dot{y}_m + \dot{y}_m \right) \]

Substituting into (1):

\[ \ddot{x}_m + 3ax_m^2 \left[ 9 + (3ax_m^2 - 2bx_m) \dot{x}_m + (6ax_m - 2b) \dot{x}_m^2 \right] \]

\[ - 2bx_m \left[ 9 + (3ax_m^2 - 2bx_m) \dot{x}_m + (6ax_m - 2b) \dot{x}_m^2 \right] = 0 \]

\[ \ddot{x}_m \left[ 1 + 3ax_m^2 \left( 3ax_m^2 - 2bx_m \right) - 2bx_m \left( 3ax_m^2 - 2bx_m \right) \right] \]

\[ + \dot{x}_m \left( 6ax_m - 2b \right) \left( 3ax_m^2 - 2bx_m \right) + \frac{1}{2}x_m \left( 3ax_m^2 - 2bx_m \right) = 0 \]
which can be further simplified to give (as in (a)):

\[ \dot{\lambda}_m = \left\{ 1 + (3 \lambda x^2 - 2 b x_n)^2 \right\} + 2 \lambda^2 \ddot{x}_m (3 \lambda x_n - 2 \dot{x}_n) + \lambda \ddot{x}_m (3 \lambda x_n - 2 \dot{x}_n) = 0 \]

(c) The Lagrange multiplier \( \lambda \) is the constraint force in the \( y \)-direction:

\[ \lambda = N \cos \theta \]

Now \( \frac{d\lambda}{dx} = \lambda \cos \theta \Rightarrow \frac{1 - \cos^2 \theta}{\cos \theta} = y^2 \]

\[ \frac{1}{\cos \theta} = y^2 + 1 \Rightarrow \cos \theta = \frac{1}{\sqrt{1 + y^2}} \]

Therefore, \( N = \frac{\lambda}{\cos \theta} = 2 \sqrt{1 + y^2} \)

where \( \lambda = m \left\{ g + (3 \lambda x^2 - 3 b x_n) \ddot{x}_m + 2 (3 \lambda x_n - 2 \dot{x}_n) \dot{x}_m \right\} \)

Note that with \( y(x) = f(x) \):

\[ \lambda = m \left\{ g + f'(x) \ddot{x}_m + f''(x) \dot{x}_m \right\} \]

and \( \sqrt{N} = m \left\{ g + f'(x) \ddot{x}_m + f''(x) \dot{x}_m \right\} \sqrt{1 + f'^2} \quad (4) \]

If \( f(x) = \sqrt{R^2 - x^2} \Rightarrow f' = \frac{-x}{\sqrt{R^2 - x^2}}, f'' = \frac{R^2}{R^2 - x^2} \) and \( \sqrt{1 + f'^2} = \frac{R}{x} \).

In polar coordinates, \( x = R \sin \theta, \dot{x}_n = R \dot{\theta} \cos \theta, \ddot{x}_m = R \ddot{\theta} \cos \theta + R \dot{\theta} \sin \theta \)

\((4) \Rightarrow \sqrt{N} = \frac{m}{\cos \theta} \left\{ g - \ddot{x}_m \left[ R \ddot{\theta} \cos \theta - R \dot{\theta}^2 \sin \theta \right] - \frac{R^2}{\cos^2 \theta} R^2 \dot{\theta} \cos \theta \right\} \]

\[ = \frac{m}{\cos \theta} \left\{ g - R \ddot{\theta} \sin \theta + R \dot{\theta}^2 \cos \theta \right\} \]
3. For a particle of mass $m$ and charge $e$ moving in a specified electric field $E(r,t)$ and magnetic field $B(r,t)$, the dynamical equations of motion are derivable from a Lagrangian of the form

$$L(r, \dot{r}, \Phi, A, t) = \frac{1}{2}m\dot{r}^2 - e\Phi(r,t) + \frac{e}{c} \cdot A(r,t).$$

The potentials used to describe the electric and magnetic fields are ambiguous up to a gauge transformation

$$A' = A + \nabla \chi, \quad \Phi' = \Phi + \frac{1}{c} \frac{\partial \chi}{\partial t},$$

where $\chi = \chi(r,t)$. Show that the Lagrangians using the two sets of potentials differ by a total time derivative (so they give the same equations of motion),

$$L'(r, \dot{r}, \Phi', A', t) = L(r, \dot{r}, \Phi, A, t) + \frac{dF}{dt},$$

and find $F$. (Optional: Using Maxwell’s equations in cgs units, show that the resulting equations of motion are consistent with the Lorentz force law, $F = e(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B})$, where $F$ is the Lorentz force.)

Find $L'(r, \dot{r}, \Phi', A', t)$:

$$L' = \frac{1}{2}m\dot{r}^2 - e\dot{\Phi}'(r,t) + \frac{e}{c} \cdot \dot{A}'(r,t)$$

$$= \frac{1}{2}m\dot{r}^2 - e\left(\dot{\Phi} - \frac{1}{c} \frac{\partial \chi}{\partial t}\right) + \frac{e}{c} \cdot \left(\dot{A} + \nabla \chi\right)$$

$$= L + \frac{e}{c} \left(\frac{\partial \chi}{\partial t} + \nabla \cdot \nabla \chi\right) = L + \frac{dF}{dt} \left(\frac{e}{c} \chi\right).$$

Therefore, $F = \frac{e}{c} \chi$. Since $L' = L + \frac{dF}{dt}$, they give the same equations of motion:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \Rightarrow \frac{d}{dt} \left(m\dot{r} + \frac{e}{c} \cdot \dot{A}\right) + e\dot{\Phi} - \frac{e}{c} \Phi(r, A) = 0$$

$$\Rightarrow m\ddot{r} + e\ddot{A} + e\dot{\Phi} - \frac{e}{c} \Phi(r, A) = 0.$$
4. A particle of charge $e$ and mass $m$ moves in a uniform magnetic field $B = B\hat{z}$.

(a) Write down the Lagrangian in a gauge where the vector potential is $A = Bx\hat{y}$.
Show that $x$, $y$, and $z$ translations are symmetries, and find the conserved quantities. (For example, $x$ translation is the transformation $x' = x + \xi$, $y' = y$, $z' = z$, and similarly for $y$ and $z$ translations.)

(b) Repeat for a different gauge in which the vector potential is $A = -By\hat{z}$.

\[ L = \frac{1}{2} m \dot{x}^2 + \frac{e}{c} \dot{z} \cdot \mathbf{A} \]

The vector potential $\mathbf{A} = Bx\hat{y}$ gives rise to $\mathbf{B} = \nabla \times \mathbf{A} = \frac{2A}{2x}$

Hence, the Lagrangian can be written as:

\[ L = \frac{1}{2} m \dot{x}^2 + \frac{e}{c} \dot{z} \cdot \left( Bx \dot{y} \right) = \frac{1}{2} m \left( \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) + \frac{e}{c} B \dot{y} \dot{z} \]

Consider an infinitesimal translation in the $x$-direction:

$\dot{x} = x$  \quad $\Rightarrow$ \quad $f_1 = 1$, $f_2 = f_3 = 0$

$\dot{y} = y$  \quad $\Rightarrow$ \quad $f_1 = f_2$, $f_3 = 1$

$\dot{z} = z$  \quad $\Rightarrow$ \quad $f_1 = f_2 = 1$, $f_3 = 0$

Find the Lagrangian in terms of the transformed coordinates:

\[ L' = L(x', y', z', \dot{x}', \dot{y}', \dot{z}') = \frac{1}{2} m \left( \dot{x}'^2 + \dot{y}'^2 + \dot{z}'^2 \right) + \frac{e}{c} B (x + \xi) \dot{y}' \]

\[ = L + \frac{e}{c} B \dot{y} \xi = L + \frac{d}{dt} \left[ \xi \left( \frac{e}{c} B \dot{y} \right) \right] \]

Hence, $L' = L + \frac{d(\xi F)}{dt}$, where $F = \frac{e}{c} B \dot{y}$. The associated conserved quantity is

\[ C = \sum_i \frac{2L}{2\dot{x}_i} f_i - F = \left( \frac{2L}{2\dot{x}} f_x + \frac{2L}{2\dot{y}} f_y + \frac{2L}{2\dot{z}} f_z \right) - \frac{e}{c} B \dot{y} \]

Hence, $C = m \dot{x} - \frac{e}{c} B \dot{y}$ is a constant of motion.

Infinitesimal translation in the $y$-direction:

$\dot{x} = x$  \quad $\Rightarrow$ \quad $f_1 = f_2 = 1$, $f_3 = 0$

$L' = L$ \quad $\Rightarrow$ \quad $F = \text{constant}$

Lagrangian in the transformed coordinates:

\[ L' = L = \text{constant} \]

Conserved quantity:

\[ C = \frac{2L}{2\dot{x}} f_x + \frac{2L}{2\dot{y}} f_y + \frac{2L}{2\dot{z}} f_z - F = \frac{2L}{2\dot{x}} = m \dot{x} \]

Note that the conserved quantity associated with this symmetry transformation is the generalized momentum corresponding to cyclic coordinate $y$: $p_y = m \dot{y} + \frac{e}{c} B \dot{x}$ (not the same as the linear momentum, $m \dot{y}$).

Infinitesimal translation in the $x$-direction:

$\dot{y} = y$  \quad $\Rightarrow$ \quad $f_1 = 1$, $f_2 = f_3 = 0$

$L' = L$ \quad $\Rightarrow$ \quad $F = \text{constant}$

Lagrangian in the transformed coordinates:

\[ L' = L = \text{constant} \]

Conserved quantity:

\[ C = \frac{2L}{2\dot{y}} f_y - F = \frac{2L}{2\dot{y}} = m \dot{y} \]

Note that $x$ is also a cyclic variable; in this case, the conserved
generalized momentum $p_z = \frac{2L}{\gamma}$ in also the z-component $q_z$
the linear momentum.

(b) In a different gauge where $\vec{A}' = -B_y \hat{x}$, $\vec{B}$ is still given
by $\vec{B} = \nabla \times \vec{A}' = -2 eBx \hat{z} = B \hat{z}$; however, the lagrangian becomes:

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{e}{c} B \dot{y} \dot{z}$$

Translation in z-direction: $L' = L \Rightarrow F = \text{constant}$

$$\Rightarrow C = \frac{2L}{\gamma} = m \dot{z} - \frac{eB \dot{z}}{c}$$

Translation in y-direction: $L' = L - \frac{e}{c} B \dot{y} \dot{x} = L + \frac{1}{c^2} \left( \frac{eB \dot{z}}{c} \right)$

$$\Rightarrow C = \frac{2L}{\gamma} + F = m \dot{y} + \frac{eB \dot{y}}{c}$$

Translation in x-direction: $L' = L \Rightarrow F = \text{constant}$

$$\Rightarrow C = \frac{2L}{\gamma} = m \dot{x}$$

The conserved quantities associated with symmetry transformations
given by translations in each of the $x$, $y$, and $z$ directions are
the same for the lagrangian written in the two gauges $\vec{A}$ and
$\vec{A}'$. [ Note that
$$\vec{A}' = \vec{A} + \nabla \chi \text{ where } \chi = -B xy.$$]