1. A particle of mass \( m \) slides without friction on the surface of a stationary inverted cone in a uniform gravitational field. The cone axis is aligned with the gravitational acceleration, \( \vec{g} \), and the cone angle is \( \alpha \).

(a) Find the Lagrangian using generalized spherical coordinates.

(b) Identify the cyclic coordinate(s) and associated conserved quantity(s).

(c) Suppose the particle has very large energy \( E \) and fixed angular momentum, \( \ell \). What are the approximate maximum and minimum possible values of the distance from the vertex, \( r \)?

(d) Find the angular speed for motion having constant \( r = r_0 \). Express your answer in terms of \( g, r_0 \) and \( \alpha \).

(e) Using the Lagrange method of undetermined multipliers to find the normal force on the particle, and verify your answer using Newton’s second law.
2. A mass $m$ is constrained to move without friction on the surface of a torus shown below. The large radius of the torus is $R$. The small radius is $r$. A point on the torus may be labeled by the coordinates $(\phi, \theta)$, as shown. $\phi$ is the angle about the $z$-axis, and $\theta$ is the angle measured about the small circle, relative to the radially outward vector in the $x - y$ plane. Suppose first, that there are no external forces (such as the force of gravity on the particle).

(a) Find the coordinates $x$, $y$ and $z$ of the mass in terms of the generalized coordinates $\phi$ and $\theta$ (and constants $r$, $R$). Write down the Lagrangian for the particle.

(b) Express the energy of the particle in terms of $\theta$, $\dot{\theta}$, and the conserved component of the angular momentum, $M_z$.

(c) Plot the “effective potential” for motion in the $\theta$-variable, and describe the motion for $M_z \neq 0$.

(d) Now gravity is turned on, $\vec{F} = -mg\hat{z}$. How is the effective potential of part (c) changed? What is the relationship between the frequency and orbital radius for uniform circular motion about the $z$-axis?