1. A particle of mass $m$ slides without friction on the surface of a stationary inverted cone in a uniform gravitational field. The cone axis is aligned with the gravitational acceleration, $g$, and the cone angle is $\alpha$.

(a) Find the Lagrangian using generalized spherical coordinates.

(b) Identify the cyclic coordinate(s) and associated conserved quantity(ies).

(c) Suppose the particle has very large energy $E$ and fixed angular momentum, $\ell$. What are the approximate maximum and minimum possible values of the distance from the vertex, $r$?

(d) Find the angular speed for motion having constant $r = r_0$. Express your answer in terms of $g$, $r_0$ and $\alpha$.

(e) Using the Lagrange method of undetermined multipliers to find the normal force on the particle, and verify your answer using Newton's second law.

(a) In spherical polar coordinates:

$$L = \frac{1}{2} m \left( r^2 + r^2 \sin^2 \theta \right) - mg r \cos \theta$$

Impose the constraint $\theta = \alpha$:

$$L = \frac{1}{2} m \left( r^2 + r^2 \sin^2 \theta \right) - mg r \cos \theta$$

(b) $L$ does not have explicit dependence on $\phi$ : \( \frac{\partial L}{\partial \dot{\phi}} \). Therefore, the associated generalized momentum is conserved:

$$P_\phi = \frac{\partial L}{\partial \dot{\phi}} = m r^2 \sin^2 \theta \dot{\phi} = \ell = \text{constant}$$

This is the $x$-component of the angular momentum, $M_x$.

Also, $L$ has no explicit time dependence: \( \frac{\partial L}{\partial t} = 0 \). Therefore, energy is conserved:

$$E = T + V = \frac{1}{2} m \dot{r}^2 + \frac{\ell^2}{2m r^2 \sin^2 \alpha} + mg r \cos \alpha$$

We can regard the motion of $r$ on the conical surface as one-dimensional motion in the coordinate $r$ under the effective potential, $V_{eff}(r)$.

(c) Plot $V_{eff}(r) = \frac{\ell^2}{2m^2 r^2 \sin^2 \alpha} + mg r \cos \alpha$.
\[ V_{\text{eff}}(r) \sim \frac{l^2}{2m^2\sin^2 \theta} \quad \text{as} \quad r \to 0 \]

\[ E \sim \frac{l^2}{2m^2\sin^2 \theta} \quad \Rightarrow \quad r_{\text{eq}} \sim \frac{1}{\sin \theta} \sqrt{2mE} \]

\[ V_{\text{eff}}(r) \sim mg r \cos \theta \quad \text{as} \quad r \to \infty \]

\[ E \sim mg r \cos \theta \quad \Rightarrow \quad r_{\text{eq}} \sim \frac{E}{mg \cos \theta} \]

(4) For circular motion at \( r = r_0 \), we must have:

\[ \left. \frac{dV_{\text{eff}}}{dr} \right|_{r = r_0} = 0 \quad \Rightarrow \quad -\frac{l^2}{m^2\sin^2 \theta} + mg \cos \theta = 0 \]

\[ r_{\text{eq}} = \frac{l^2}{m^2 \sin^2 \theta} \]

or

\[ l^2 = m^2 g^2 \sin^2 \theta \cos \theta \cos \theta \]

The angular speed is given by:

\[ \omega^2 = \frac{l^2}{(m^2 \sin^2 \theta)^2} = \frac{mg^2 \sin^2 \theta \cos^2 \theta}{m^2 \sin^2 \theta} = \frac{g \cos \theta}{r_0 \sin^2 \theta} \]

Therefore, \( \phi = \left( \frac{g \cos \theta}{r_0 \sin^2 \theta} \right)^{\frac{1}{2}} \).

(5) We will write the constraint that the mass moves on the conical surface in two ways; both shall give the constraint force, \( \vec{N} = -N \hat{\theta} \).

1) Spherical polar coordinates

\[ \vec{L} = \frac{1}{2} m (v^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) - mg r \cos \theta + \lambda (\theta - \theta_0) \]

\[ \dot{r} = m \ddot{r} - mr \sin \theta \dot{\theta}^2 + mg \cos \theta = 0 \quad (c) \]

\[ \dot{\theta} = \frac{d}{dt} (m^2 \dot{\theta}) - m^2 \sin \theta \dot{\phi}^2 - mg \sin \theta + \lambda = 0 \quad (d) \]

\[ \dot{\phi} = \frac{d}{dt} (m^2 \sin^2 \theta \dot{\phi}) = 0 \quad (e) \]

\[ \lambda = \theta - \theta_0 = 0 \quad (f) \]

(4) \( \Rightarrow \) \( \theta = \theta_0 \) \( \Rightarrow \) \( \dot{\theta} = 0 \).

(5) \( \Rightarrow \) \( \lambda = m \dot{r} \sin \theta \cos \theta \dot{\phi}^2 + mg \sin \theta \dot{\phi} \)

\[ \lambda = r \left( m \dot{r} \sin \theta \cos \theta \dot{\phi}^2 + mg \sin \theta \right) \]

\( \lambda \) has dimensions of torque (length \( \times \) force). It is the constraint torque that keeps \( \theta \) fixed at \( \theta = \theta_0 \). Since

\[ \vec{r}_n = \vec{r} \times \vec{N} = -\hat{r} \times N \hat{\theta} = -r N \hat{\phi} \]
Therefore, \( N = mrsin\alpha \cos \phi \hat{x} + mg \sin \alpha \).

Let's use Newton's laws to arrive at this result:

\[
\vec{F} = r \sin \alpha \cos \phi \hat{x} + r \sin \alpha \sin \phi \hat{y} + r \cos \alpha \hat{z}
\]

\[
\vec{F} = \vec{F}_m + \vec{F}_g
\]

\[
\vec{F} = \vec{F}_m + \vec{F}_g = N \hat{\phi} + mg \sin \alpha \hat{y}
\]

\[
\vec{F} = \vec{F}_m + \vec{F}_g = N \hat{\phi} + mg \sin \alpha \hat{y}
\]

\[
\Rightarrow \vec{F} = \vec{F}_m + \vec{F}_g = N \hat{\phi} + mg \sin \alpha \hat{y}
\]

If \( \phi \neq 0 \), the net torque on \( m \) is nonzero. However, it has no component in the \( z \)-direction (\( mg \) in the \( x-y \) plane), consistent with conservation of \( \phi \) and \( P_y = 0 \).

2) Cartesian coordinates:

\[
\vec{r} = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = mg \hat{z} + \lambda (z + \alpha x \sqrt{x^2 + y^2})
\]

where we have written the constraint equation as:

\[
\hat{g}(x,y,z) = \lambda = \cot \alpha \sqrt{x^2 + y^2}
\]

Note that \( \hat{g} \) has dimensions of length and written in this form, \( \lambda \) will be the constraint force in the \( z \)-direction, \( N_z \).

\[
x = N_x + \alpha \dot{x} \frac{\dot{z}}{\sqrt{x^2 + y^2}} = 0 \tag{1'}
\]

\[
y = mg + \alpha \dot{y} \frac{\lambda}{\sqrt{x^2 + y^2}} = 0 \tag{2'}
\]

\[
z = \dot{z} + mg - \lambda = 0 \tag{3'}
\]

[Note: The net torque on \( m \) (about the origin, given by the cone vertex) is: \( \vec{\tau} = \vec{\tau}_N + \vec{\tau}_g \).]
2. A mass $m$ is constrained to move without friction on the surface of a torus shown below. The large radius of the torus is $R$. The small radius is $r$. A point on the torus may be labeled by the coordinates $(\phi, \theta)$, as shown. $\phi$ is the angle about the $z$-axis, and $\theta$ is the angle measured about the small circle, relative to the radially outward vector in the $x-y$ plane. Suppose first, that there are no external forces (such as the force of gravity on the particle).

![Diagram of a torus with labeled coordinates and vectors]

(a) Find the coordinates $x$, $y$, and $z$ of the mass in terms of the generalized coordinates $\phi$ and $\theta$ (and constants $r, R$). Write down the Lagrangian for the particle.

(b) Express the energy of the particle in terms of $\theta, \phi$, and the conserved component of the angular momentum, $M_R$.

(c) Plot the "effective potential" for motion in the $\theta$ variable, and describe the motion for $M_R \neq 0$.

(d) Now gravity is turned on, $\vec{F} = -mg\hat{z}$. How is the effective potential of part (c) changed? What is the relationship between the frequency and orbital radius for uniform circular motion about the $z$-axis?
(a) \[ x = (R + r \cos \theta) \cos \phi \quad \text{and} \quad y = (R + r \cos \theta) \sin \phi \]
\[ z = r \sin \theta \quad \text{as} \quad 0 \leq \phi < 2\pi, \ 0 \leq \theta < 2\pi\]

\[ \dot{x} = -r \dot{\theta} \sin \theta \cos \phi - (R + r \cos \theta) \sin \phi \sin \theta \]
\[ \dot{y} = r \dot{\theta} \sin \theta \sin \phi + (R + r \cos \theta) \cos \phi \sin \theta \]
\[ \dot{z} = r \dot{\theta} \cos \theta \]

In the absence of gravity,

\[ L = \frac{1}{2} m \left( \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) = \frac{1}{2} m \left[ \dot{x}^2 + (R + r \cos \theta)^2 \dot{\phi}^2 \right] \]

(b) \( \phi \) is a cyclic variable, therefore \( p_\phi = \frac{L_\phi}{\dot{\phi}} = \lambda \) is constant.

\[ L = m (R + r \cos \theta)^2 \dot{\phi} \]

This is just the \( z \)-component of angular momentum:

\[ M_z = m (z \dot{x} - y \dot{z}) = m (R + r \cos \theta)^2 \dot{\phi} \]

The total energy of \( m \) is given by

\[ E = T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m (R + r \cos \theta)^2 \dot{\phi}^2 \]

\[ = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} \frac{\dot{z}^2}{(R + r \cos \theta)^2} \]

\[ = \frac{1}{2} m \dot{x}^2 + V_{eff}(\theta) \]

Therefore, we can think of the motion of \( m \) on the toroidal surface as one-dimensional motion in the coordinate \( \theta \) under \( V_{eff}(\theta) \).

(c) \[ V_{eff}(\theta) = \frac{1}{2m(R-r)^2} \]

To plot \( V_{eff}(\theta) \), first look at possible maxima/minima:

\[ \frac{dV_{eff}}{d\theta} = \frac{r \dot{\theta}}{m (R + r \cos \theta)^3} \left[ \cos \theta - \frac{3}{2} \sin \theta \right] \]

\[ \frac{d^2V_{eff}}{d\theta^2} = \frac{r \dot{\theta}^2}{m (R + r \cos \theta)^3} \left[ \cos \theta + 2 \sin \theta \right] \]

\[ \frac{d^2V_{eff}}{d\theta^2} = 0 \Rightarrow \sin \theta = 0 \text{ or } \theta = 0, \pi \]

At \( \theta = 0 \), \( \frac{dV_{eff}}{d\theta} = \frac{r \dot{\theta}}{m (R-r)^3} > 0 \Rightarrow V_{eff} \text{ minimum at } \theta = 0 \)

\[ \frac{dV_{eff}}{d\theta} = \frac{r \dot{\theta}}{m (R-r)^3} < 0 \Rightarrow \theta = \pi \text{ maximum at } \theta = \pi \]

Hence, \( V_{eff}(\theta) \) looks like plot above.

The different types of motion will depend on the value of \( V_{eff}(\theta) \).

For \( l \neq 0 \):

\[ V(\theta) > \frac{\dot{z}^2}{2m(R-r)^2} \]

All values of \( \theta \) are accessible. Since the periods of \( \theta \) and \( \phi \) motions are not commensurate in general, the orbits are not closed in general.
(ii) \( \frac{l^2}{2m(R+r)^2} < E < \frac{l^2}{2m(R-r)^2} \)

\( \Theta \) will be bound: \( \Theta \in [-\Theta_0, \Theta_0] \) where

\[ V_{\text{eff}}(\Theta) = E \Rightarrow \frac{l^2}{2m(R+r\cos\Theta)^2} = E \Rightarrow \cos\Theta = \frac{l}{\sqrt{2mE}} - R \]

For example, for \( \Theta_0 < E \), the orbit looks like:

(iii) Special cases of (ii)

\[ E = \frac{l^2}{2m(R+r)^2} = V_{\text{eff}}(\Theta_0) \Rightarrow \Theta_0 = 0 \]

Circular motion about outer diameter, \( R+r \)

\[ E = \frac{l^2}{2m(R-r)^2} = V_{\text{eff}}(\Theta_0) \Rightarrow \Theta_0 = \pm \pi \]

Circular motion about inner diameter, \( R-r \)

Stability of the circular orbit is determined by the sign of \( d^2V_{\text{eff}}/d\Theta^2 \): circular motion at \( \Theta = 0 \) is stable and at \( \Theta = \pm \pi \) is unstable. \( V_{\text{eff}}(0) > 0 \) \( V_{\text{eff}}(\pm \pi) < 0 \)

With gravity turned on, the motion is no longer symmetric about \( \Theta = 0 \); the apeirogon (\( \Theta \rightarrow -\Theta \)) symmetry of the problem is broken by \( F = -mg \sin \Theta \). Find the new effective potential:

\[ E = T + V = \frac{1}{2}m r^2 \dot{\Theta}^2 + \frac{l^2}{2m(R+r\cos\Theta)^2} + mg r \sin\Theta \]

The new effective potential is:

\[ \tilde{V}_{\text{eff}}(\Theta) = \frac{l^2}{2m(R+r\cos\Theta)^2} + mg r \sin\Theta \]

To plot it, determine maxima/minima:

\[ \frac{d\tilde{V}_{\text{eff}}}{d\Theta} = \frac{dV_{\text{eff}}}{d\Theta} + mg r \cos\Theta \Rightarrow \frac{d\tilde{V}_{\text{eff}}}{d\Theta} \bigg|_{\Theta = 0} = mg r > 0 \]

Therefore, the minimum of the new effective potential with gravity turned on is shifted to the left.
Circular motion as achieved for \( \theta = \Theta^* \):

\[
\frac{d^2 \psi}{d\theta} + \frac{m g \cos \Theta^*}{R + r \cos \Theta^*} = 0
\]

\[
\Rightarrow \quad \frac{1}{m} \frac{d^2 \psi}{d\theta^2} + \frac{m g \cos \Theta^*}{R + r \cos \Theta^*} = 0
\]

\[
\Rightarrow \quad \int^2 = - \frac{m^2 \cot \Theta^*}{(R + r \cos \Theta^*)^3} > 0 \quad < 0 \text{ since } \Theta^* < 0
\]

Now, \( \phi = \frac{L}{m (R + r \cos \Theta^*)} = \sqrt{\frac{m g \cot \Theta^*}{R + r \cos \Theta^*}} \)

The orbital radius is given by \( R_{orb} = R + r \cos \Theta^* \)

\[
\Rightarrow \quad \cos \Theta^* = \frac{R_{orb} - R}{r}
\]

\[
\Rightarrow \quad \tan \Theta^* = \sqrt{\frac{1}{\cos \Theta^* - 1}} = \frac{\sqrt{r^2 - (R_{orb} - R)^2}}{(R_{orb} - R)}
\]

\[
\cot \Theta^* = \frac{R_{orb} - R}{\sqrt{r^2 - (R_{orb} - R)^2}}
\]

Therefore, \( \phi = \left[ \frac{g (R - R_{orb})}{\sqrt{r^2 - (R_{orb} - R)^2}} \right]^{1/2} \).