“The physicist needs a facility in looking at problems from several points of view ... [An] intuitive understanding is a completely unmathematical, imprecise, and inexact thing, but it is absolutely necessary for a physicist.”
- - - Richard Feynman

1. **In Case of Charged Oscillators, Use the Ladders**

A charged particle $e$ is trapped in a 1-D SHO potential that is immersed in a uniform electric field $\mathcal{E}$. In this case, we pick up an extra potential energy term of the form $\mathcal{H}' = e\mathcal{E}x$. Based on this, calculate the shift in the energy level of the $n^{th}$ state to second order in the “small” quantity $(e\mathcal{E})$. Along the way, show that the first-order correction vanishes for all $n$. *Hint: use raising and lowering operators, rather than working in coordinate space.*

2. **Rutherford was Right! The Nucleus is Not a Point**

Consider a single electron in the electric field of a nucleus of charge $Ze$ and radius $r_0$. If the charge is uniformly distributed over the spherical nucleus, then Gauss’ Law tells us that classically the electron has an electrostatic potential energy of (with $k \equiv 1/4\pi\varepsilon_0$):

$$V(r) = \begin{cases} k\frac{Ze^2}{2r_0} \left( \frac{r^2}{r_0^2} - 3 \right) & r \leq r_0 \\ -k\frac{Ze^2}{r} & r > r_0 \end{cases}$$

(a) Defining the perturbation to be the *difference* between $V(r)$ above and the potential energy due to a point charge, show that the first-order energy shift is given by

$$\Delta E_{n\ell} = k\frac{Ze^2}{2r_0} \int_0^{r_0} \left[ R_{n\ell}(r) \right]^2 \left( \frac{r^2}{r_0^2} + \frac{2r_0}{r} - 3 \right) r^2 dr$$

(b) With the approximation $R_{n\ell}(r) \approx R_{n\ell}(0)$ inside the nucleus, show that your expression from part a reduces to

$$\Delta E_{n\ell} = k e^2 \frac{2}{5} \frac{r_0^2}{n^3 a_0^2} \frac{Z^4}{\delta_{\ell 0}} \delta_{\ell 0}$$

Explain why the assumption made for $R_{n\ell}(r)$ is reasonable, and why the $\delta$-function in the answer is just what you might expect.
3. De Generalization of De Generacy

In class, we argued that the first-order corrections to the energies of $d$ degenerate states are given by the eigenvalues of the matrix $H'_{ij}$, and the eigenvectors give us the “correct” set of states in the degenerate sub-space. These statements were based on working out the $d = 2$ case explicitly and then generalizing the results in an “obvious” way. *Prove* that these claims are true by considering a set of $d$ degenerate states, $\{\psi^0_j\}$, that obey

$$ H^0 \psi^0_j = E^0 \psi^0_j \quad \text{with} \quad \langle \psi^0_i | \psi^0_j \rangle = \delta_{ij} \quad i, j = 1, 2, 3, \ldots, d $$

In analogy with Eq. 6.17 in the text, consider the linear combinations

$$ \psi^0 = \sum_{j=1}^{d} \alpha_j \psi^0_j $$

Now follow the steps in section 6.2.1 to arrive at the generalized form of Eq. 6.22:

$$ \sum_{i=1}^{d} H'_j \alpha_i = E^1 \alpha_j \quad \text{where} \quad H'_j \equiv \langle \psi^0_j | H' | \psi^0_i \rangle $$

4. Welcome 2-D SHO!

Using non-degenerate perturbation theory, we worked out a few problems for the 1-D SHO. In two dimensions, however, we encounter a system where most excited states are degenerate in energy, so more sophisticated techniques are needed. With this as a warning:

Consider a 2-D isotropic SHO, which in Cartesian coordinates has the Hamiltonian

$$ \hat{H}^0 = -\hbar^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{2} m \omega^2 (x^2 + y^2) $$

where $\omega = (\kappa/m)^{1/2}$ with $\kappa =$ spring constant. The eigenvalues of this Hamiltonian are

$$ E_n^{(0)} = \hbar \omega (n + 1) $$

with $n = n_x + n_y$, and $n_x, n_y = 0, 1, 2, \ldots$.

Now turn on an external potential of the form $\hat{H}' = \varepsilon xy$. Treating this as a small quantity, calculate the first-order energy shift of the ground state and show that it vanishes.

5. The (2-D) SHO Must Go On!

Continuing the above: solve for the first-order energy shift of the first excited state(s).