Generalized momenta and the Hamiltonian

Let’s define generalized momentum (canonical momentum):

\[ p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma} \]

for independent generalized coordinates

Lagrange’s equations can be written as:

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0 \quad \sigma = 1, \ldots, n \]

if the lagrangian does not depend on some coordinate,

\[ \frac{\partial L}{\partial q_\sigma} = 0. \]

related to the symmetry of the problem - the system is invariant under some continuous transformation. For each such symmetry operation there is a conserved quantity!

the corresponding momentum is a constant of the motion, a conserved quantity.
Three-dimensional motion in a one-dimensional potential:

\[ L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(z) \]

x and y are cyclic coordinates - shift symmetry

corresponding generalized momenta:

\[ p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \]

are conserved:

\[ \dot{p}_x = \dot{p}_y = 0 \]

conservation of linear momentum

Three-dimensional motion in a one-dimensional potential:

\[ L = \frac{1}{2}m(\dot{r}^2 + r^2 \dot{\phi}^2) - V(r) \]

\( \phi \) is a cyclic coordinates - rotational symmetry

corresponding generalized momentum:

\[ p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2 \dot{\phi} \]

is conserved:

\[ \frac{d}{dt} mr^2 \dot{\phi} = 0 \]

conservation of angular momentum
If the lagrangian does not depend explicitly on the time, then the hamiltonian is a constant of the motion:

\[ H \equiv \sum_{\sigma} p_{\sigma} \dot{q}_{\sigma} - L \]

**Proof:**

\[
\frac{dH}{dt} = \sum_{\sigma} \left( p_{\sigma} \frac{d\dot{q}_{\sigma}}{dt} + \dot{q}_{\sigma} d\dot{p}_{\sigma} - \frac{\partial L}{\partial q_{\sigma}} \frac{d\dot{q}_{\sigma}}{dt} - \frac{\partial L}{\partial \dot{q}_{\sigma}} \frac{d\dot{q}_{\sigma}}{dt} \right) = \sum_{\sigma} \dot{q}_{\sigma} \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_{\sigma}} - \frac{\partial L}{\partial q_{\sigma}} \right) = 0
\]

\[ p_{\sigma} = \frac{\partial L}{\partial \dot{q}_{\sigma}} \]

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_{\sigma}} - \frac{\partial L}{\partial q_{\sigma}} = 0 \quad \sigma = 1, \ldots, n \]
If there are only time-independent potentials and time-independent constraints, then the Hamiltonian represents the total energy.

Proof:

\[ H = \sum_{\sigma} p_\sigma \dot{q}_\sigma - L \]

\[ \sum_{\sigma} p_\sigma \dot{q}_\sigma = \sum_{\sigma} \frac{\partial L}{\partial \dot{q}_\sigma} \dot{q}_\sigma = \sum_{\sigma} \frac{\partial T}{\partial q_\sigma} \dot{q}_\sigma = 2T \]

\[ p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma} \]

\[ T = \frac{1}{2} \sum_i m_i \dot{x}_i^2 = \frac{1}{2} \sum_{\sigma} \sum_\lambda \left( \sum_i m_i \frac{\partial x_i}{\partial q_\sigma} \frac{\partial x_i}{\partial q_\lambda} \right) \dot{q}_\sigma \dot{q}_\lambda = \frac{1}{2} \sum_{\sigma} \sum_\lambda m_{\sigma \lambda} (q) \dot{q}_\sigma \dot{q}_\lambda \]

\[ \dot{x}_i = \sum_{\sigma} \frac{\partial x_i}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial x_i}{\partial t} \]

\[ m_{\sigma \lambda} = m_{\lambda \sigma} = m_{\sigma \lambda}^* = m_{\lambda \sigma}^* \]

\[ p_\sigma = \frac{\partial T}{\partial \dot{q}_\sigma} = \frac{1}{2} \sum_\lambda m_{\sigma \lambda} \dot{q}_\lambda + \frac{1}{2} \sum_\lambda m_{\lambda \sigma} \dot{q}_\lambda = \sum_\lambda m_{\sigma \lambda} \dot{q}_\lambda \]

\[ H = 2T - (T - V) = T + V = E \]

\[ \frac{\partial L}{\partial t} = 0 \quad \text{and} \quad \frac{\partial x_i}{\partial t} = 0 \]

\[ H = T + V = E = \text{const} \]
Bead on a Rotating Wire Hoop: 

hoop rotates with constant angular velocity about an axis perpendicular to the plane of the hoop and passing through the edge of the hoop. No friction, no gravity.

\[ \theta \text{ is the generalized coordinate} \]

\[ x = a \cos \omega t + a \cos (\omega t + \theta) \]
\[ y = a \sin \omega t + a \sin (\omega t + \theta) \]

\[ \dot{x} = -a \omega \sin \omega t - a(\omega + \dot{\theta}) \sin (\omega t + \theta) \]
\[ \dot{y} = a \omega \cos \omega t + a(\omega + \dot{\theta}) \cos (\omega t + \theta) \]

\[ L = T - V \]

\[ T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) \]

\[ \cos \omega t \cos (\omega t + \theta) + \sin \omega t \sin (\omega t + \theta) = \cos (\omega t + \theta - \omega t) = \cos \theta \]

\[ T = L = \frac{1}{2} ma^2 [\omega^2 + (\omega + \dot{\theta})^2 + 2\omega(\omega + \dot{\theta}) \cos \theta] \]

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0 \quad \sigma = 1, ..., n-k \]

\[ \ddot{\theta} + \omega^2 \sin \theta = 0 \]

pendulum equation
Bead on a Rotating Wire Hoop: 

- hoop rotates with constant angular velocity about an axis perpendicular to the plane of the hoop and passing through the edge of the hoop.
- No friction, no gravity.

\[ \theta \text{ is the generalized coordinate} \]

\[ T = L = \frac{1}{2}ma^2[\omega^2 + (\omega + \dot{\theta})^2 + 2\omega(\omega + \dot{\theta})\cos \theta] \]

**generalized momentum:**

\[ p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ma^2(\omega + \dot{\theta}) + ma^2\omega \cos \theta \]

**the hamiltonian:**

\[ H = p_\theta \dot{\theta} - L = ma^2[\frac{1}{2}\dot{\theta}^2 - \omega^2(1 + \cos \theta)] \]

\[ \frac{\partial L}{\partial t} = 0 \Rightarrow \frac{dH}{dt} = 0 \]

is a constant of the motion, but it does not represent the total energy!

\[ \frac{\partial x_i}{\partial t} \neq 0 \]