We will consider small-amplitude oscillations of mechanical systems about static equilibrium, e.g. coupled pendulums:

Applications: vibrations of molecules, crystals,...

Consider a system described by a set of $n$ independent generalized coordinates, with time-independent potential and no time-varying constraints:

\[
L = L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n) \\
V = V(q_1, \ldots, q_n) \\
x_i = x_i(q_1, \ldots, q_n)
\]

Equations of motion:

\[
\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} = Q_\sigma = -\frac{\partial V}{\partial q_\sigma} \quad \sigma = 1, \ldots, n
\]

Static equilibrium:

\[
q_\sigma = q_\sigma^0 \quad \dot{q}_\sigma = \ddot{q}_\sigma = 0 \quad \sigma = 1, \ldots, n
\]

All generalized forces have to vanish:

\[
Q_\sigma = -\left(\frac{\partial V}{\partial q_\sigma}\right)_{q_\sigma^0} = 0 \quad \sigma = 1, \ldots, n
\]

Stability:

If the potential has a minimum, the equilibrium is stable
Consider a small displacement from equilibrium:

\[ q_\sigma = q^0_\sigma + \eta_\sigma \quad \sigma = 1, \ldots, n \]

\[ \dot{q}_\sigma = \ddot{\eta}_\sigma \quad \sigma = 1, \ldots, n \]

**Kinetic energy:**

\[
T = \frac{1}{2} \sum_\sigma \sum_\lambda m_{\sigma \lambda} \ddot{\eta}_\sigma \ddot{\eta}_\lambda
\]

\[
m_{\sigma \lambda} = m_{\lambda \sigma} = m^*_\sigma = m^*_\lambda
\]

evaluated at the equilibrium - constant matrix!

**Potential energy:**

\[
V(q_1, \ldots, q_n) = V(q^0_1, \ldots, q^0_n) + \sum_\sigma \eta_\sigma \left( \frac{\partial V}{\partial q^0_\sigma} \right)_{q^0} + \frac{1}{2} \sum_\sigma \sum_\lambda \eta_\sigma \eta_\lambda \left( \frac{\partial^2 V}{\partial q^0_\sigma \partial q^0_\lambda} \right)_{q^0}
\]

\[ q_\sigma = -\left( \frac{\partial V}{\partial q^0_\sigma} \right)_{q^0} = 0 \quad \sigma = 1, \ldots, n \]

\[ v_{\sigma \lambda} = \frac{\partial^2 V}{\partial q^0_\sigma \partial q^0_\lambda} \]

evaluated at the equilibrium - constant matrix!

**Lagrangian:**

\[ q_\sigma = q^0_\sigma + \eta_\sigma \quad \sigma = 1, \ldots, n \]

\[
L = T - V = \frac{1}{2} \sum_\sigma \sum_\lambda (m_{\sigma \lambda} \ddot{\eta}_\lambda - v_{\sigma \lambda} \eta_\sigma) - V_0
\]

quadratic in small displacements and derivatives

\[ m_{\sigma \lambda} = m_{\lambda \sigma} = m^*_\sigma = m^*_\lambda \]

constant real symmetric matrices

**Equations of motion:**

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0 \quad \sigma = 1, \ldots, n
\]

\[
\sum_\lambda (m_{\sigma \lambda} \ddot{\eta}_\lambda + v_{\sigma \lambda} \dot{\eta}_\lambda) = 0 \quad \sigma = 1, \ldots, n
\]

linear in small displacements and derivatives
Normal modes - pendulum

Small displacement problem for a system described by 1 generalized coordinate:

\[ m_{\sigma,k} = m \]
\[ v_{\sigma,k} = k \]

Introduce complex coordinate:

\[ z = \frac{k}{m} \]
\[ \eta = \Re z \]

We seek a solution of the form:

\[ z = z^0 e^{i\omega t} \]

Eigenvalue equation:

\[ (\omega^2 - \frac{k}{m}) z^0 = 0 \]
\[ \omega = \pm \omega_1 \]
\[ \omega_1 = \left( \frac{k}{m} \right)^{1/2} \]

General solution:

\[ z(t) = z^{(1)} e^{i\omega_1 t} + z^{(1)*} e^{-i\omega_1 t} \]

General solution of the original problem:

\[ \eta = \Re z = \frac{1}{2} (z + z^*) = \frac{1}{2} [z^{(1)} + (z^{(1)})^*] e^{i\omega_1 t} + \text{complex conjugate} \]

\[ z^{(1)} + (z^{(1)})^* = z^{(1)} = \rho^{(1)} e^{i\phi_1} \]

\[ \eta = \rho^{(1)} \cos (\omega_1 t + \phi_1) = \Re (z^{(1)} e^{i\omega_1 t}) = \Re (\rho^{(1)} e^{i(\omega_1 t + \phi_1)}) \]

We wrote the general solution using only positive eigenvalue
Normal modes - general - Part 1

In general, we need to solve a set of $n$ linear homogeneous coupled differential equations with constant coefficients. It is convenient to introduce complex parameters:

$$\sum_{\alpha=1}^{n} (m_{\alpha} \ddot{z}_{\alpha} + v_{\alpha} z_{\alpha}) = 0 \quad \sigma = 1, \ldots, n$$

$$\eta_{\sigma} = \Re e z_{\sigma} \quad \sigma = 1, \ldots, n$$

We will look first for normal modes:

$$z_{\sigma} = z_{0\sigma} e^{i\omega t} \quad \sigma = 1, \ldots, n$$

all the coordinate oscillate with the same frequency

$n$ linear homogeneous coupled algebraic equations

Coupled pendulums - Part 1

Small-amplitude oscillations coupled pendulums:

$$\eta_{l} = \sin \theta \approx \theta$$

$$T = \frac{1}{2} m[\dot{\left( \theta_{1} \right)}^{2} + \dot{\left( \theta_{2} \right)}^{2}] \approx \frac{1}{2} m(\dot{\eta}_{1}^{2} + \dot{\eta}_{2}^{2})$$

$$V_{spring} + V_{gravity} \approx \frac{k}{2} (\eta_{1} - \eta_{2})^{2} + \frac{mg}{2l} (\eta_{1}^{2} + \eta_{2}^{2})$$

$$d - d_{0} = \eta_{2} - \eta_{1} + O(\eta^{4})$$

Height raised $= l(1 - \cos \theta) \approx \frac{1}{2} \omega^{2} = \frac{\eta_{1}^{2}}{2l}$

length of the spring - natural length

Lagrangian:

$$L = \frac{1}{2} m(\dot{\eta}_{1}^{2} + \dot{\eta}_{2}^{2}) - \frac{mg}{2l} (\eta_{1}^{2} + \eta_{2}^{2}) - \frac{k}{2} (\eta_{1}^{2} + \eta_{2}^{2} - 2\eta_{1}\eta_{2})$$
Langrange’s equations:

\[ L = \frac{1}{2} m (\ddot{\eta}_1^2 + \ddot{\eta}_2^2) - \frac{mg}{2l} (\dot{\eta}_1^2 + \dot{\eta}_2^2) - \frac{1}{2} k (\dot{\eta}_1^2 + \dot{\eta}_2^2 - 2\eta_1 \eta_2) \]

\[
\begin{align*}
\ddot{\eta}_1 + \left( k + \frac{mg}{l} \right) \eta_1 - k \eta_2 &= 0 \\
\ddot{\eta}_2 + \left( k + \frac{mg}{l} \right) \eta_2 - k \eta_1 &= 0
\end{align*}
\]

Solving for normal modes:

This is the general solution (we will prove it later)

\[ \eta_\sigma = C_{\sigma \rho} \cos (\omega t + \phi) \quad \sigma = 1, 2 \]

both pendulums oscillate with the same frequency

Linear equations - math review:

Consider a set of \( n \) linear inhomogeneous equations (real coefficients):

\[
\begin{align*}
a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &= y_1 \\
a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n &= y_2 \\
&\quad \cdots \cdots \cdots \\
a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n &= y_n
\end{align*}
\]

\[ \sum_{j=1}^{n} a_{ij} x_j = y_i \quad i = 1, \ldots, n \]

Solution:

If \( \det \mathbf{a} \neq 0 \) then
Consider a set of $n$ linear homogeneous equations:

$$\sum_{j=1}^{n} a_{ij} x_j = y_i \quad i = 1, \ldots, n$$

with all $y_i = 0$

Solution:

- If $\det a \neq 0$ then 
  
  only trivial solution

- If $\det a = 0$ then 
  
  at least one equation is linearly dependent, and can be discarded. Then, assuming the $n$-th component of $x$ is non-zero, we can divide all remaining equations by it... and obtain a set of $n-1$ inhomogeneous equations.

Back to coupled pendulums:

$$(v - m\omega^2)\rho = 0$$

has a non-trivial solution only if:

$$ax = 0$$

$$\det a = 0$$

two solutions for normal-mode frequencies:

$$\omega_1 = \left(\frac{g}{l}\right)^{1/2}$$

free pendulum

$$\omega_2 = \left(\frac{g}{l} + \frac{k}{m}\right)^{1/2}$$

higher frequency
Corresponding normal-mode eigenvectors:

\[
\begin{align*}
\omega_1 &= \left(\frac{g}{l}\right)^{1/2} \\
\omega_2 &= \left(\frac{g}{l} + \frac{k}{m}\right)^{1/2} \\
k\rho_1^{(1)} - k\rho_2^{(1)} &= 0 \\
\rho_1^{(1)} &= +\rho_2^{(1)} \\
- k\rho_1^{(2)} - k\rho_2^{(2)} &= 0 \\
\rho_1^{(2)} &= -\rho_2^{(2)} \\
\eta_\sigma &= C_\sigma \rho_\sigma \cos(\omega t + \phi) \quad \sigma = 1, 2
\end{align*}
\]

Normal modes - general - Part 2

Our set of linear homogeneous equations has a nontrivial solution only if:

\[
\det |v_{\sigma\lambda} - \omega^2 m_{\sigma\lambda}| = 0
\]

this leads to an n-th order polynomial, which has n roots:

\[
\omega^2 = \omega_s^2 \\
s = 1, 2, \ldots, n
\]

all the roots are real

\[
\sum_{\sigma=1}^{n} (v_{\sigma\lambda} - \omega_s^2 m_{\sigma\lambda}) z_{\lambda}^{(s)} = 0 \\
\sigma = 1, \ldots, n
\]

given non-trivial solution

\[
\omega_s^2 = \sum_{\sigma=1}^{n} \sum_{\lambda=1}^{n} z_{\lambda}^{(s)} v_{\sigma\lambda} \delta_{\lambda s} \\
\sum_{\sigma=1}^{n} \sum_{\lambda=1}^{n} z_{\lambda}^{(s)} v_{\sigma\lambda} m_{\sigma\lambda} = m_{s\lambda}
\]

\[
(v_{\sigma\lambda})^* = v_{\lambda\sigma}^* \quad s = 1, \ldots, n
\]

differs only by interchange of dummy summation indices
Stability:

- **ω₁² < 0** unstable
  - imaginary frequency leads to runaway solutions
- **ω₁² ≥ 0** stable
  - positive-definite if the potential is a minimum at equilibrium
  - positive-definite, since m is the mass matrix

Form of the general solution:

\[ \sum_{\sigma=1}^{n} (\omega_{\sigma} - \omega_{1}^2 m_{\sigma_{1}}) z_{\sigma}^{(0)} = 0 \quad \sigma = 1, \ldots, n \]

- n-1 ratios of components \( z_{\sigma}/z_{\sigma_{1}} \) are real, there can be only one overall phase
- one component can be chosen arbitrarily

Form of the general solution (continued):

s-th eigenvalue:

\[ \sum_{\sigma} \rho_{\sigma}^{(s)} = \omega_{s}^{2} \sum_{\sigma} m_{\sigma_{1}} \rho_{\sigma}^{(s)} \quad \sigma = 1, \ldots, n \]

\[ \sum_{\sigma} \rho_{\sigma}^{(0)} \]

\[ (\omega_{s}^{2} - \omega_{1}^{2}) \sum_{\sigma} \sum_{\sigma} \rho_{\sigma}^{(s)} m_{\sigma_{1}} \rho_{\sigma}^{(t)} = 0 \]

For non-degenerate eigenvalues:

\[ \sum_{\sigma} \sum_{\sigma} \rho_{\sigma}^{(s)} m_{\sigma_{1}} \rho_{\sigma}^{(t)} = 0 \quad s \neq t \]

we can normalize the eigenvectors according to

and write the general solution as:

\[ z_{\sigma}^{(s)} = C^{(s)} e^{i\phi_{\sigma}} \rho_{\sigma}^{(s)} \quad \sigma = 1, \ldots, n \]

for degenerate eigenvalues we can use Gram-Schmidt orthogonalization procedure to get