Consider a system on \(N\) particles with all their relative separations fixed: it has 3 translational and 3 rotational degrees of freedom.

Motion with One Arbitrary Fixed Point:

Motion with No Fixed Point:

Angular momentum:

\[
\mathbf{L} = \sum_p m_p \mathbf{r}_p \times \dot{\mathbf{r}}_p = \sum_p m_p [\mathbf{r}_p \times (\mathbf{\omega} \times \mathbf{r}_p)] = \sum_p m_p [\mathbf{r}_p^2 \mathbf{\omega} - (\mathbf{\omega} \cdot \mathbf{r}_p) \mathbf{r}_p]
\]

continuum limit:

\[
\mathbf{L} = \int d^3r \rho(r) \mathbf{r} (r^2 \mathbf{\omega} - (\mathbf{\omega} \cdot \mathbf{r}) \mathbf{r})
\]

using the inertia tensor it can be written as:

\[
L_{ij} = \frac{1}{2} \int d^3r \rho(r) \delta_{ij} r^2 - x_i x_j
\]

and it is related to the kinetic energy:

\[
T = \frac{1}{2} \sum_{i,j} L_{ij} \omega_i \omega_j = \mathbf{L} \cdot \mathbf{\omega}
\]

Kinetic Energy:

\[
T = \frac{1}{2} \sum_p m_p v_p^2 = \frac{1}{2} \sum_p m_p (\mathbf{\omega} \times \mathbf{r}_p) \cdot (\mathbf{\omega} \times \mathbf{r}_p)
\]

continuum limit:

\[
T = \frac{1}{2} \int d^3r \rho(r) \mathbf{\omega}^2 r^2 - (\mathbf{\omega} \cdot \mathbf{r})^2
\]

it is convenient to define the inertia tensor:

\[
I_{ij} = \int d^3r \rho(r) (\delta_{ij} r^2 - x_i x_j)
\]

in a body-fixed frame it is constant and depends only on the mass distribution

the kinetic energy has a simple form:

\[
T = \frac{1}{2} \sum_{i,j} I_{ij} \omega_i \omega_j
\]

both \(I\) and \(\mathbf{\omega}\) depend on the choice of the coordinate system, but \(T\) does not!
Total force:

\[ F_i = F_i^e + \sum_{j \neq i} F_{ji} \]

Total momentum:

\[ P = \sum_i m_i \mathbf{r}_i = \sum_i m_i v_i = \mathbf{MR} = \mathbf{MV} \]

Total angular momentum:

\[ \mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i \]

The total angular momentum:

\[ \mathbf{L} = \sum_{i} m_i \mathbf{r}_i \times \mathbf{v}_i = \mathbf{L}_{\text{cm}} + \mathbf{L}' \]

Changes in \( \mathbf{L} \) and \( \mathbf{P} \) arise only from external forces!

Working in the center of mass coordinate system:

\[ \mathbf{r}_i = \mathbf{R} + \mathbf{r}_i' \]

Total angular momentum:

\[ \mathbf{L} = \sum_i m_i (\mathbf{r}_i + \mathbf{r}_i') \times (\mathbf{v}_i + \mathbf{v}_i') \]

Changes in \( \mathbf{L} \) and \( \mathbf{P} \) arise only from external forces!
The total kinetic energy:

\[ T = \frac{1}{2} \sum_i m_i v_i^2 \]

\[ T_{\text{cm}} = \frac{1}{2} M V^2 \]

motion of the center of mass

internal motion about the CM

For conservative forces:

\[ F_i = F_{i0} + \sum_{j \neq i} F_{ij} \]

\[ F_{ij} = -\nabla_j V(r_{ij}) = -\nabla_j V(r_i - r_j) \]

Assume isotropic potential, force lies along \( r_i - r_j \)

Inertia tensor

We have to learn how to evaluate the inertia tensor in a body-fixed frame. First, let's define the inertia tensor in a body-fixed frame with the origin at the center of mass:

\[ I_{ij} = \int d^3 \rho (r(x))^2 - x_i x_j \]

Then the inertia tensor in a body-fixed frame with the origin displaced by \( \mathbf{a} \) is given by:

\[ I_{ij} = I_{ij} + M (a^2 \delta_{ij} - a_i a_j) \]

Review:

\[ I = \sum_i m_i \mathbf{r}_i \times (\mathbf{v}_i \times \mathbf{r}_i) \]

Formulas for \( T' \) and \( L' \) are identical to the case of a motion with one fixed point!

Example (a uniform disk of radius \( a \)):

\[ I_{ij} = I_{ij} + M (a^2 \delta_{ij} - a_i a_j) \]
**Principal axes** (a coordinate system in which the inertia tensor is diagonal):

\[ I_{ij} = I_i \delta_{ij} \]

all the formulas simplify in such a coordinate system

Diagonalizing a real symmetric matrix:

\[
\sum_{j=1}^{3} I_{ij} e_j = 2 \delta_i \\
\sum_{j=1}^{3} (I_{ij} - 2 \delta_{ij}) e_j = 0
\]

similar to solving for normal modes, \( \nu \rightarrow 1 \) and \( m \rightarrow 1 \)

non-trivial solution only if:

\[
\det |I_{ij} - \lambda e_{ij}| = 0
\]

we get 3 eigenvalues, 3 eigenvectors, and we can form the modal matrix:

\[
\mathbf{a} = \begin{bmatrix} e^{(1)} & e^{(2)} & e^{(3)} \end{bmatrix}
\]

\[
\mathbf{a}^T \mathbf{a} = \mathbf{a}^T \mathbf{a}^{-1} \mathbf{a} = 1
\]

new orthonormal basis - principal axes

**Angular Momentum in the new coordinate system:**

\[
\mathbf{I}_{\text{new}} = \mathbf{a}^T \mathbf{I} \mathbf{a} = \begin{bmatrix} I_{1}^{(1)} & 0 \\ 0 & I_{2}^{(2)} & 0 \\ 0 & 0 & I_{3}^{(3)} \end{bmatrix}
\]

\[
\mathbf{I}_{\text{new}} = \mathbf{a}^T \mathbf{I} \mathbf{a} = \mathbf{a}^T \mathbf{a} \mathbf{a} = \begin{bmatrix} I_{1}^{(1)} \mathbf{e}^{(1)} \\ I_{2}^{(2)} \mathbf{e}^{(2)} \\ I_{3}^{(3)} \mathbf{e}^{(3)} \end{bmatrix}
\]

**Principal moments of inertia:**

\[
I_{ij} = \int d^3r \, \rho(r) \left( r^2 - (r \cdot \mathbf{e}^{(i)})^2 \right)
\]

\[
\mathbf{a} = \begin{bmatrix} e^{(1)} & e^{(2)} & e^{(3)} \end{bmatrix}
\]

projections of \( \mathbf{r} \) along principal axes

**Modal matrix diagonalizes the inertia tensor:**

\[
\mathbf{a}^{-1} \mathbf{I} \mathbf{a} = \mathbf{I}_{\text{new}} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}
\]

we can define a new set of angular velocities:

\[
\omega = \mathbf{a}^{-1} \omega \quad \xi = \mathbf{a}^{-1} \xi
\]

\[
\begin{bmatrix} \xi \end{bmatrix}^T = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \mathbf{a}^{-1} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}
\]

projections of \( \omega \) along the principal axes

**Euler’s Equations**

**Describing rotational motion:**

based on FW-27

valid in any inertial frame but also in the cm frame:

\[
\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}_{\text{torque}} = \begin{bmatrix} \sum_{j=1}^{3} \epsilon_{ij} \omega_j \end{bmatrix}_{\text{cm}}
\]

**Euler’s equations:**

\[
L_i = I_i \omega_i
\]

projecting onto the principal body axes

\[
(dL_i/dt)_{\text{body}} = \dot{L}_i + \omega \times L = \Gamma_i
\]

simple formulas, but limited use, since both angular velocities and torques are evaluated in a time dependent body-fixed frame (principal axes);

used mostly for torque-free motion or partially constrained motion (examples follow)
Applications - Compound Pendulum

**a rigid body constrained to rotate about a fixed axis:** based on FW-28

we can choose the coordinate system so that:

\[ \mathbf{R} \equiv \mathbf{i} \mathbf{e}_1 \]

\[ \dot{\phi} = \omega \mathbf{e}_2 \]

one degree of freedom

ignoring friction, the only torque comes from gravity:

\[ \Gamma_\phi = -Mg \sin \phi \]

\[ \Gamma_\parallel = \sum_i m_i \mathbf{r}_i \times \mathbf{g} = M \mathbf{r}_c \times \mathbf{g} \]

in both inertial and body frame

\[ \mathbf{r}_c = \mathbf{r}_0 \]

\[ \mathbf{R}_c = \mathbf{R}_0 \]

\[ l = \frac{1}{2} \sum_i \mathbf{r}_i^2 \]

\[ l = l_{33} = \int d^3 \rho \mathbf{x}(x^2 - x_3^2) = \int d^3 \rho \mathbf{x}(x_1^2 + x_2^2) = \int d^3 \rho \mathbf{x}( \rho(\mathbf{e}_1 \mathbf{e}_3) ) \]

**equation of motion of an arbitrary compound pendulum:**

\[ I \ddot{\phi} = -Mgl \sin \phi \]

\[ I = l_{33} = \int d^3 \rho \mathbf{x}( \rho^2 ) \]

for small oscillations, the motion is simple harmonic:

\[ \ddot{\phi} = -\Omega^2 \phi \]

\[ \Omega = \sqrt{\frac{l}{l + l_1}} \]

Let’s rewrite it in terms of the moment of inertia about the axis going through the center of mass:

\[ I = I_{33} = I_{33} + Ml^2 \]

\[ I_{33} = Mk^2 \]

\[ l = M(k^2 + l^2) = Mk^2 \]

\[ k = (k^2 + l^2)^{1/2} \]

radius of gyration about the axis through Q

\[ \Omega_\phi = \frac{gl}{k^2 + l^2} = \frac{g}{l_1} \]

\[ l_1 = l + \frac{l^2}{I} \]

**Response of a baseball bat to a transverse force:**

(we neglect gravity)

**motion of the center of mass (N-2nd):**

\[ M \dot{\mathbf{R}} = \mathbf{f} - f_c \]

\[ \mathbf{R} \cdot \dot{\mathbf{e}}_2 = \mathbf{L} \]

the only direction R can move due to constraints

**torque equation:**

\[ M \dot{\phi} = f - f_c \]

\[ \mathbf{R} \cdot \dot{\mathbf{e}}_2 = \mathbf{L} \]

\[ M \ddot{\phi} = f - f_c \]

\[ f_c = \frac{l_1 - l}{l} \]

\[ f_c = \frac{l_1 - l}{l} \]

if the force is applied at the center of percussion

the reaction force at the point of support vanishes!

if applied beyond that point, the reaction force has the same direction as applied force