Action-Angle Variables

Hamilton-Jacobi theory can be used to calculate frequencies of various motions without completely solving the problem if the motion of the system is both **separable** and **periodic**.

$$W(q_1, \ldots, q_n, x_1, \ldots, x_n) = W_1(q_1, x_1, \ldots, x_n) + \cdots + W_n(q_n, x_1, \ldots, x_n)$$

**libration**

**rotation**

e.g. harmonic oscillator
e.g. pendulum going over the top

Let’s define the **action variables** of the system:

$$J_\sigma = J_\sigma(x_1, \ldots, x_n) \quad \sigma = 1, \ldots, n$$

**they are constants of the motion:**

$$J_\sigma = \frac{1}{2} p_\sigma dq_\sigma \quad \sigma = 1, \ldots, n$$

we assume the relations are invertible

$$x_\sigma = x_\sigma(J_1, \ldots, J_n) \quad \sigma = 1, \ldots, n$$

we already know one constant:

$$x_1 = x_1(J_1, \ldots, J_n) = H(J_1, \ldots, J_n) = E$$

we can use action variables as the integration constants as in $S$:

$$W(q_1, \ldots, q_n, x_1(J), \ldots, x_n(J)) = \tilde{W}(q_1, \ldots, q_n, J_1, \ldots, J_n)$$

$$S(q_1, \ldots, q_n, x_1(J), \ldots, x_n(J); t) = \tilde{S}(q_1, \ldots, q_n, J_1, \ldots, J_n; t) = W - x_1(J)t$$

This generates following transformation:

$$p_\sigma = \frac{\partial S}{\partial q_\sigma} \quad \sigma = 1, \ldots, n$$

$$\dot{q}_\sigma = \frac{\partial S}{\partial J_\sigma}$$

$S$ satisfies Hamilton-Jacobi equation:

$$\dot{H} \equiv 0$$

Let’s define the **angle variables** of the system:

$$w_\sigma = \frac{\partial S}{\partial x_\sigma} \quad \sigma = 1, \ldots, n$$

$$\dot{x}_\sigma = \frac{\partial S}{\partial J_\sigma}$$

We will need the differential of the angle variables:

$$dw_\sigma = \sum_\sigma \frac{\partial \tilde{W}}{\partial q_\sigma} dq_\sigma = \frac{\partial \tilde{W}}{\partial J_\sigma} \sum_\sigma \frac{\partial \tilde{W}}{\partial q_\sigma} dq_\sigma = \frac{\partial \tilde{W}}{\partial J_\sigma} \sum_\sigma p_\sigma dq_\sigma$$

“frequency”, a constant of the motion

the angle variables increase linearly with time

depends on coordinates and constants definition separability
If the motion of the system is periodic, then in one period of the entire system

\[ \Delta t = n_i \tau_i, \quad \lambda = 1, \ldots, n \]

all the degrees of freedom execute some integral number of individual periods.

The corresponding change in the angle variables:

\[ \Delta w_i = \sum \frac{\partial}{\partial J_i} J_i = \sum \frac{\partial}{\partial J_i} n_i \tau_i = n_i \]

but also

\[ \Delta w = v_{\tau} \Delta t = v_{\tau} n_i \tau_i \]

fundamental frequencies of the system:

\[ v_{\tau} \tau_i = 1, \quad \lambda = 1, \ldots, n \]

indeed it is frequency

Example (harmonic oscillator in two dimensions with different spring constants):

The action variables are then:

\[ \frac{1}{2} p_x^2 + \frac{1}{2m} p_y^2 + \frac{1}{2k_1} q_1^2 + \frac{1}{2k_2} q_2^2 = \alpha \]

\[ W_i(q_i, \dot{q}_i) = W_1(q_1) + W_2(q_2) \]

separability

\[ \frac{1}{2m} (\frac{\partial W_1}{\partial q_1})^2 + \frac{1}{2m} (\frac{\partial W_2}{\partial q_2})^2 + \frac{1}{2k_1} q_1^2 + \frac{1}{2k_2} q_2^2 = \alpha \]

\[ \frac{1}{2m} \left( \frac{1}{2m} \frac{\partial W_1}{\partial q_1} \right)^2 + \frac{1}{2k_1} q_1^2 = \alpha_1 \]

\[ \frac{1}{2m} \left( \frac{1}{2m} \frac{\partial W_2}{\partial q_2} \right)^2 + \frac{1}{2k_2} q_2^2 = \alpha_2 \]

Poisson brackets

Let’s define the Poisson bracket of two functions, \( F \) and \( G \):

\[ [F, G]_{PB} = \sum \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \]

it is obviously antisymmetric

\[ [F, G]_{PB} = -[G, F]_{PB} \]

Poisson bracket of a function \( F \) and the Hamiltonian:

\[ [H, F]_{PB} = \sum \frac{\partial H}{\partial p_i} \frac{\partial F}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial F}{\partial p_i} \]

\[ = -\sum \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial H}{\partial p_i} \]

Hamilton’s equations:

\[ \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \sigma = 1, \ldots, n \]

\[ \dot{q}_i = \frac{\partial H}{\partial p_i} \]

Poisson bracket formulation of classical mechanics!
PB is equivalent formulations of classical mechanics:

Hamilton’s equations:

\[
\begin{align*}
\dot{q}_i &= -\{H, q_i\}_\text{PB} = \frac{\delta H}{\delta p_i} \\
\dot{p}_i &= -\{H, p_i\}_\text{PB} = \frac{\delta H}{\delta q_i}
\end{align*}
\]

\[\{H, H\}_\text{PB} = 0\]

PB of coordinates and canonical momenta:

\[\{p_s, q_b\}_\text{PB} = -\delta_{sb}\]

\[\{p_s, p_b\}_\text{PB} = \{q_s, q_b\}_\text{PB} = 0\]

Canonical transformation to a new set of coordinates and momenta, Q and P, preserves Hamilton equations and thus Poisson brackets. A canonical transformation can be defined as one that preserves Poisson-bracket description of mechanics.

Transition to quantum mechanics:

Define the commutator of two quantities:

\[\{A, B\} = AB - BA\]

Canonical quantization prescription:

\[\{A, B\}_\text{PB} \rightarrow \frac{1}{i\hbar}\{A, B\}\]

non-commuting operators acting on a Hilbert space

Equation of motion:

\[\frac{d\hat{F}}{dt} = -[\hat{H}, \hat{F}] + \frac{\delta \hat{F}}{\delta \hat{q}}\]

Heisenberg operator equation of motion