Central Forces

Motion of a particle with mass $m$ subject to a central force:

$$\nabla \times F = 0$$

$$\nabla \times (f\vec{v}) = (\nabla f) \times \vec{v} + f(\nabla \times \vec{v})$$

no torque, $\mathbf{r} \times \mathbf{F} = 0$ $\implies \mathbf{l} = \mathbf{r} \times \mathbf{p} = \text{const.}$

(motion is planar, perpendicular to $\mathbf{l}$)

It is convenient to work with polar coordinates:

$$x = r \cos \phi \quad \text{and} \quad y = r \sin \phi$$
Conservation of energy:

\[ E = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + V(r) \]

\[ E = T + V = \frac{1}{2}m(r^2 + r^2\dot{\phi}^2) + V(r) \]

(first integral
(involves only 1st time derivatives)
(homework, problem 1.6)

Conservation of angular momentum:

\[ \mathbf{l} = \mathbf{r} \times \mathbf{p} = \text{const.} \]

\[ l_z = xp_y - yp_x = m(x\dot{y} - y\dot{x}) \]

\[ l = mr^2\dot{\phi} \]

motion in a small interval of time \( dt \):

\[ d\phi = \dot{\phi} \, dt \]

\[ dA = \frac{1}{2}r(r \, d\phi) = \frac{1}{2}r^2\dot{\phi} \, dt = (l/2m) \, dt \]

\[ \dot{A} = \frac{l}{2m} = \text{const} \]
Kepler’s second law:

An imaginary line drawn from each planet to the Sun sweeps out equal areas in equal times.

(a simple consequence of the conservation of the angular momentum for central forces)
Effective potential:

\[ E = \frac{1}{2} m \dot{r}^2 + V_{\text{eff}}(r) \]

Analyzing the motion:

The angular momentum piece acts as a repulsive potential, **centrifugal barrier**

\[ l = m r^2 \dot{\phi} \]

One dimensional potential:

\[ V_{\text{eff}}(r) = V(r) + \frac{l^2}{2mr^2} \]

\[ E = T + V = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) + V(r) \]

Turning points:

\[ E - V_{\text{eff}} = \frac{1}{2} m \dot{r}^2 \geq 0 \]

Minimum energy, circular orbit:

\[ E \geq V_{\text{eff}} \]

\[ E = V_{\text{eff}}(r_0) \]

\[ V(r) = -\lambda/r \]
Solution to the trajectory:

\[ E = \frac{1}{2} m r^2 + V_{\text{eff}}(r) \]

\[ V_{\text{eff}}(r) = V(r) + \frac{\mu^2}{2m r^2} \]

\[ l = mr^2 \dot{\phi} \]

\[
    t = \pm \left( \frac{l}{m} \right)^{1/2} \int dr \left[ E - V_{\text{eff}}(r) \right]^{-1/2} + t_0
\]

invert to find \( r(t) \)

\[
    \phi = \ln^{-1} \int dt \left[ r(t) \right]^{-2} + \phi_0
\]

plug in \( r(t) \) to find \( \phi(t) \)

In practice this is often too complicated, but we can always find the geometric orbit \( r(\phi) \) (at least numerically):

\[
    \dot{r} = \frac{dr}{d\phi} \dot{\phi} = \frac{dr}{d\phi} \left( \frac{l}{m r^2} \right)
\]

\[
    l = mr^2 \dot{\phi}
\]

invert to find \( r(\phi) \)
Kepler’s problem (planetary motion):

Special case when the central force is gravitational force:

from the general solution:

\[ V(r) = -\frac{m\gamma}{r} \]

\[ \gamma \equiv GM \]

\[ u = r^{-1} \]

\[ \phi = \phi_0 + \int du \left( \frac{2mE}{l^2} + \frac{2m^2\gamma u}{l^2} - u^2 \right)^{-1/2} \]

we find:

\[ \phi = \phi_0 \pm \arccos \left( 1 - ul^2/m^2\gamma \right) \left( 1 + 2El^2/m^3\gamma^2 \right)^{1/2} \]

\[ \int \frac{dx}{\sqrt{ax^2 + bx + c}} = \begin{cases} \frac{1}{\sqrt{a}} \ln \left( 2\sqrt{a} \sqrt{ax^2 + bx + c} + 2ax + b \right) \\ -\frac{1}{\sqrt{-a}} \sinh^{-1} \left( \frac{2ax + b}{\sqrt{b^2 - 4ac}} \right) \end{cases} \]

or

\[ \frac{1}{\sqrt{a}} \sinh^{-1} \left( \frac{2ax + b}{\sqrt{4ac - b^2}} \right) \]

\[ \arccos x = \frac{\pi}{2} - \arcsin x \]
Orbit solution:

For negative energy, orbits are bound and represent an ellipse.
Conic sections:

- Circle
- Ellipse
- Parabola
- Hyperbola

<table>
<thead>
<tr>
<th>Conic section</th>
<th>Equation</th>
<th>Eccentricity (e)</th>
</tr>
</thead>
<tbody>
<tr>
<td>circle</td>
<td>$x^2 + y^2 = a^2$</td>
<td>0</td>
</tr>
<tr>
<td>ellipse</td>
<td>$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$</td>
<td>$\sqrt{1 - \frac{b^2}{a^2}}$</td>
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<tr>
<td>parabola</td>
<td>$y^2 = 4ax$</td>
<td>1</td>
</tr>
<tr>
<td>hyperbola</td>
<td>$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$</td>
<td>$\sqrt{1 + \frac{b^2}{a^2}}$</td>
</tr>
</tbody>
</table>
Ellipse:

constant total distance from two foci located at x=+f and x=-f:

\[ d + d' = 2a \]

eccentricity

\[ f = \epsilon a \]

in polar coordinates with the origin at the left focus:

\[ r(1 - \epsilon \cos \phi) = a(1 - \epsilon^2) \]

follows from the law of cosines:

\[ d^2 = d'^2 - 4df \cos \phi + 4f^2 \]

\[ c^2 = a^2 + b^2 - 2ab \cos \gamma \]
Orbit solution:

\[ r^{-1} = C(1 - \epsilon \cos \phi) \]

\[ \epsilon = \left(1 + \frac{2El^2}{m^2\gamma^2}\right)^{1/2} \]
\[ C = \frac{m^2\gamma}{l^2} \]
\[ \gamma \equiv GM \]

ellipse in polar coordinates with the origin at the left focus:

\[ r(1 - \epsilon \cos \phi) = a(1 - \epsilon^2) \]

Kepler’s first law:
The orbit of a particle with negative energy in a gravitational field is an ellipse with the center at one focus.

\[ \epsilon = \epsilon = \sqrt{\frac{a^2 - b^2}{a^2}} = \sqrt{1 - \left(\frac{b}{a}\right)^2} = f/a \]
\[ f = \sqrt{a^2 - b^2} \]

energy of an orbit does not depend on angular momentum
Kepler’s third law:
The square of an orbital period is proportional to the cube of a.

Kepler’s 2nd law

\[ A = \pi ab = \int_0^t dt \dot{A} = \frac{l}{2m} \]

Kepler’s 3rd law

\[ A = \pi a^2 (1 - e^2)^{1/2} = \pi a^2 \left( \frac{2|E|l^2}{m^3 \gamma^2} \right)^{1/2} \]

\[ e = \sqrt{\frac{a^2 - b^2}{a^2}} \]

\[ E = -\frac{m\gamma}{2a} \]

\[ \tau = 2\pi a^{3/2} \gamma^{-1/2} \]

\[ \gamma \equiv GM \]

Planetary orbit eccentricities
- Mercury: .206
- Venus: .0068
- Earth: .0167
- Mars: .0934
- Jupiter: .0485
- Saturn: .0556
- Uranus: .0472
- Neptune: .0086
- Pluto: .25
Comment on general central potential:

orbits typically not closed:

\[ \Delta \phi = l(2m)^{-1/2} \int_{r_{\min}}^{r_{\max}} dr \ r^{-2} [E - V_{\text{eff}}(r)]^{-1/2} \]

if rational multiple of \( \pi \) then orbit is closed, otherwise it is open

Harmonic oscillator:

\[ V(r) \propto r^2 \]

Orbit is an ellipse with the origin at the center!

(homework, problem 1.10)