Cross section:

incident flux
(number of particles crossing unit transverse area per unit time)

differential cross section
(constant of proportionality)

\[
\text{Number of events of type } p \text{ per unit time} = F \, d\sigma_p
\]

event \( p \) = elastic scattering through a deflection angle between \( \theta \) and \( \theta + d\theta \):

\[
F \, 2\pi b \, db = F \, d\sigma_{el}(\theta) = F \left( \frac{d\sigma}{d\Omega} \right)_{el} \, d\Omega
\]

differential cross section:
(per unit solid angle)

\[
\left( \frac{d\sigma}{d\Omega} \right)_{el} = \frac{b}{\sin \theta} \left| \frac{db}{d\theta} \right|
\]

total cross section:

\[
\sigma_p = \int d\Omega \left( \frac{d\sigma}{d\Omega} \right)_{el}
\]
Rutherford scattering:
(the scattering of a particles with charge $ze$ and mass $m$ from a nucleus with charge $Ze$ and mass $M$)

Newton’s second law: $\frac{\mu \ddot{r}}{r^3} = \frac{zZe^2 r}{\mu}$ — Coulomb’s force

the same as for Kepler problem, hyperbolic gravitational orbits with the replacement:

**Orbit solution:**

$$r^{-1} = C(1 - \epsilon \cos \phi)$$

$$\epsilon = \left(1 + \frac{2E\ell^2}{\mu^3 \gamma^2}\right)^{1/2}$$

$$C = \frac{\mu^2 \gamma}{\ell^2}$$

for $C<0$ (repulsive potential):

$$r^{-1} = |C|(\epsilon \cos \phi - 1)$$

is the right branch of hyperbola in polar coordinates with the origin at the left focus.
Hyperbola:

constant difference in distance from two foci located at \(x=+f\) and \(x=-f\):

- Eccentricity: \(\varepsilon = \frac{f}{a} > 1\)
- Asymptotes of the two branches:
  \[
  \cos \alpha = \varepsilon^{-1}
  \]

Right branch in polar coordinates with the origin at the right focus:

\[
r (1 - \varepsilon \cos \phi) = a(\varepsilon^2 - 1)
\]

follows from the law of cosines:

\[
d'^2 = d^2 + 4f^2 - 4df \cos(\pi - \phi)
\]

Right branch in polar coordinates with the origin at the left focus:

\[
r (\varepsilon \cos \phi - 1) = a(\varepsilon^2 - 1)
\]

follows from the law of cosines:

\[
d^2 = d'^2 + 4f^2 - 4d'f \cos \phi
\]
for positive $C$ we found:

$$\cot \frac{1}{2} \theta = \frac{v^2_{\infty} b}{\gamma}$$

$b$ is identical for both signs of the potential

relation between $b$ and the scattering angle:

$$b = \frac{|zZ| e^2}{\mu v^2_{\infty}} \cot \frac{1}{2} \theta$$

Rutherford formula:

$$\left( \frac{d\sigma}{d\Omega} \right)_{el} = \frac{b}{\sin \theta} \left| \frac{db}{d\theta} \right|$$

$$E = \frac{1}{2} \mu v^2_{\infty}$$

$$\left( \frac{d\sigma}{d\Omega} \right)_{el} = \left( \frac{zZe^2}{2\mu v^2_{\infty}} \right)^2 \frac{1}{\sin^4 \frac{1}{2} \theta} = \left( \frac{zZe^2}{4E \sin^2 \frac{1}{2} \theta} \right)^2$$

even the total cross section is divergent for small scattering angles!

What does it mean?
Scattering by a hard sphere:

\[ \theta = 2\Phi_m - \pi \]

\[ b = a \sin \Phi_m \]

\[ b = a \cos \frac{1}{2}\theta \quad \text{for} \quad b < a \]

\[ \theta = 0 \quad \text{for} \quad b > a \]

**differential cross section:**

\[ \left( \frac{d\sigma}{d\Omega} \right)_{el} = \frac{b}{\sin \theta} \left| \frac{db}{d\theta} \right| \]

\[ \left( \frac{d\sigma}{d\Omega} \right)_{el} = \frac{1}{4}a^2 \]

**total cross section:**

\[ \sigma_T = \int d\Omega \left( \frac{d\sigma}{d\Omega} \right)_{el} = \pi a^2 \]

Alternatively:

\[ F 2\pi b \, db = F d\sigma_{el}(\theta) = F \left( \frac{d\sigma}{d\Omega} \right)_{el} \, d\Omega \]

\[ \sigma_T = 2\pi \int_0^{b_{max}} b \, db = \pi b_{max}^2 \]

\[ \theta = 0 \quad \text{for} \quad b > a \]

Infinite for Rutherford-like scattering

Cross section is a geometric area (circle)