Constrained motion and generalized coordinates

Often, the motion of particles is restricted by constraints, and we want to:
- work only with independent degrees of freedom (coordinates)
- eliminate forces of constraint

Motion of $N$ particles, $n = 3N$ degrees of freedom, subject to $k$ equations relating coordinates:

\[ f_j(x_1, \ldots, x_n, t) = c_j \quad j = 1, 2, \ldots, k \]

can be time dependent

holonomic constraints

the system has $n - k = 3N - k$ degrees of freedom!

Generalized coordinates:

- cartesian coordinates
- subject to $k$ constraints
- any set of $n - k = 3N - k$ independent coordinates that completely specify the system

change of a cartesian coordinate induced from changes in generalized coordinates in $dt$:

\[
dx_i = \frac{\partial x_i}{\partial q_1} dq_1 + \cdots + \frac{\partial x_i}{\partial q_{n-k}} dq_{n-k} + \frac{\partial x_i}{\partial t} dt
\]

or, in a compact way:

\[
dx_i = \sum_{s=1}^{n-k} \frac{\partial x_i}{\partial q_s} dq_s + \frac{\partial x_i}{\partial t} dt \quad i = 1, \ldots, n
\]
Virtual displacements:

\[ \delta x_i = \sum_{\sigma = 1}^{n-k} \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma \quad i = 1, ..., n \]

are defined as infinitesimal, instantaneous displacements of the coordinates consistent with the constraints.

\[ x_1 = x_1(q_1, q_2, ..., q_{n-k}, t) \]
\[ \vdots \]
\[ x_n = x_n(q_1, q_2, ..., q_{n-k}, t) \]

\[ d\dot{x}_i = \sum_{\sigma = 1}^{n-k} \frac{\partial x_i}{\partial q_\sigma} d\dot{q}_\sigma + \frac{\partial x_i}{\partial t} dt \quad i = 1, ..., n \]

D’Alembert’s Principle

The forces of constraint do no work under a virtual displacement:

The forces of constraint are perpendicular to the direction of motion and thus they do no work.

We can rewrite Newton’s 2nd law as:

\[ \dot{p}_i = F_i^{(a)} + R_i \quad i = 1, ..., n \]

or

\[ \sum_i (F_i^{(a)} + R_i - \dot{p}_i) \delta x_i = 0 \]

\[ \sum_i R_i \delta x_i = 0 \]

D’Alembert’s principle:

\[ \sum_i (F_i^{(a)} - \dot{p}_i) \delta x_i = 0 \]

forces of constraint have disappeared!
Lagrange’s equations

We want to rewrite D’Alembert’s principle in terms of the generalized coordinates.

\[ \sum_i (F_i^a - \dot{p}_i) \delta x_i = 0 \]

The applied force piece:

\[ \delta W = \sum_i F_i \delta x_i = \sum_{\sigma=1}^{n-k} \left( \sum_{i=1}^{n} F_i \frac{\partial x_i}{\partial q_\sigma} \right) \delta q_\sigma = \sum_{\sigma=1}^{n-k} Q_\sigma \delta q_\sigma \]

The virtual work done by applied forces under virtual displacement.

\[ \delta x_i = \sum_{\sigma=1}^{n-k} \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma \]

\[ Q_\sigma = \sum_{i=1}^{n} F_i \frac{\partial x_i}{\partial q_\sigma} \]

generalized forces

can be calculated directly from this definition, or from computing the virtual work done by applied forces for virtual displacement along a given generalized coordinate.

Juggling with time derivatives:

\[ \frac{dx_i}{dt} = \dot{x}_i = \sum_{\sigma} \left( \frac{\partial x_i}{\partial q_\sigma} \right) \dot{q}_\sigma + \left( \frac{\partial x_i}{\partial t} \right) \]

\[ \dot{x}_i = \dot{x}_i(q_1, \ldots, q_{n-k}; \dot{q}_1, \ldots, \dot{q}_{n-k}, t) \]

only functions of \( q \) and \( t \)

function of generalized coordinates, generalized velocities and time - all physically independent variables!

(useful formulas:

\[ \frac{\partial \dot{x}_i}{\partial q_\sigma} = \frac{\partial x_i}{\partial q_\sigma} \]

other variables are kept constant when taking partial derivatives

\[ \frac{d}{dt} \left( \frac{\partial x_i}{\partial q_\sigma} \right) = \left( \frac{\partial x_i}{\partial q_\sigma} \right) \frac{d}{dt} \]

the order of partial derivatives can be interchanged

(if specified at a given time, subsequent motion of the system is determined)
The momentum piece:

\[ \sum_l \dot{p}_l \delta x_l = \sum_l m_l \ddot{x}_l \delta x_l = \sum_l \left( \sum_i m_i \ddot{x}_i \right) \delta q_\sigma \]

\[ \dot{p}_l = m_l \ddot{x}_l \]

\[ \delta x_l = \sum_{\sigma=1}^{n-k} \frac{\delta \dot{x}_l}{\delta q_\sigma} \delta q_\sigma \]

\[ \sum_l m_l \ddot{x}_l \frac{\delta x_l}{\delta q_\sigma} = \sum_l m_l \left( \frac{d}{dt} \left( \frac{\delta x_l}{\delta q_\sigma} \right) - \frac{d}{dt} \frac{\delta \dot{x}_l}{\delta q_\sigma} \right) = \sum_l m_l \left( \frac{d}{dt} \frac{\delta x_l}{\delta q_\sigma} \right) - \frac{d}{dt} \left( \frac{\delta \dot{x}_l}{\delta q_\sigma} \right) \]

\[ \frac{d}{dt} \left( \frac{\delta x_l}{\delta q_\sigma} \right) = \frac{\partial}{\partial q_\sigma} \left( \frac{\dot{x}_l}{2} \right) \]

\[ \sum_l m_l \ddot{x}_l \frac{\delta x_l}{\delta q_\sigma} = \frac{\partial}{\partial q_\sigma} \left( \frac{1}{2} \sum_i m_i \ddot{x}_l^2 \right) \]

**kinetic energy:**

\[ T = \frac{1}{2} \sum_l m_l \dot{x}_l^2 = T(q_1, \ldots, q_{n-k}, \dot{q}_1, \ldots, \dot{q}_{n-k}; t) \]

Now we can rewrite D’Alembert’s principle in terms of the generalized coordinates:

\[ \sum_l \left( F_{i(g)} - \dot{p}_l \right) \delta x_l = 0 \]

\[ \delta W = \sum_l F_l \delta x_l = \sum_{l=1}^{n-k} \left( \sum_{i=1}^{n} F_l \frac{\delta x_l}{\delta q_\sigma} \right) \delta q_\sigma = \sum_{l=1}^{n-k} Q_\sigma \delta q_\sigma \]

\[ \sum_l \dot{p}_l \delta x_l = \sum_{l=1}^{n-k} \left( \frac{d}{dt} \frac{\delta T}{\delta q_\sigma} - \frac{\partial T}{\partial q_\sigma} \right) \delta q_\sigma \]

\[ T = \frac{1}{2} \sum_l m_l \dot{x}_l^2 = T(q_1, \ldots, q_{n-k}; \dot{q}_1, \ldots, \dot{q}_{n-k}; t) \]

**Lagrange’s equations:**

\[ \frac{d}{dt} \frac{\delta T}{\delta q_\sigma} - \frac{\partial T}{\partial q_\sigma} = Q_\sigma \quad \sigma = 1, \ldots, n-k \]

**all q are independent and arbitrary**

**n-k equations for n-k independent generalized coordinates**

( equivalent to Newton’s 2nd law)
For conservative forces (potential energy depends only on the position):

\[ V(x_1, \ldots, x_n) = V(q_1, \ldots, q_{n-k}, t) \]

\[ \frac{\partial V}{\partial q_\sigma} = 0 \]

generalized forces are given by negative gradients with respect to corresponding generalized coordinate:

\[ Q_\sigma = \sum_i F_i \frac{\partial x_i}{\partial q_\sigma} = -\sum_i \left[ \frac{\partial}{\partial x_i} V(x_1, \ldots, x_n) \right] \frac{\partial x_i}{\partial q_\sigma} \]

\[ = -\frac{\partial}{\partial q_\sigma} V(q_1, \ldots, q_{n-k}, t) \]

Lagrange’s equations for conservative forces:

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0 \quad \sigma = 1, \ldots, n-k \]

\[ L \equiv T - V \]

Lagrangian