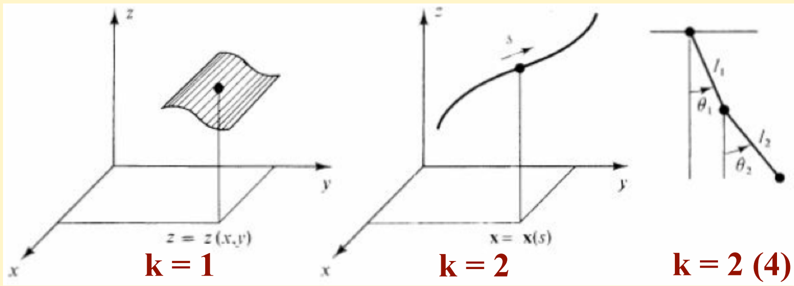
Constrained motion and generalized coordinates

based on FW-13

Often, the motion of particles is restricted by constraints,



constraints correspond to forces

and we want to:

- **work only with independent degrees of freedom (coordinates)**
- **eliminate forces of constraint**

Motion of N particles, n = 3N degrees of freedom, subject to k equations relating coordinates:

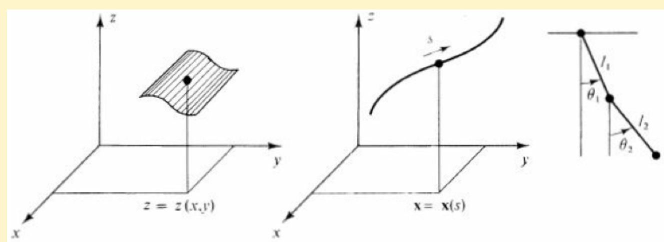
$$f_j(x_1, \ldots, x_n, t) = c_j \qquad j = 1, 2, \ldots, k$$

can be time dependent

holonomic constraints

the system has n - k = 3N - k degrees of freedom!

Generalized coordinates:



$$x_1 = x_1(q_1, q_2, \ldots, q_{n-k}, t)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$x_n = x_n(q_1, q_2, \ldots, q_{n-k}, t)$$

cartesian coordinates subject to k constraints

any set of n - k = 3N - k independent coordinates that completely specify the system

change of a cartesian coordinate induced from changes in generalized coordinates in *dt*:

$$dx_1 = \frac{\partial x_1}{\partial q_1} dq_1 + \cdots + \frac{\partial x_1}{\partial q_{n-k}} dq_{n-k} + \frac{\partial x_1}{\partial t} dt$$

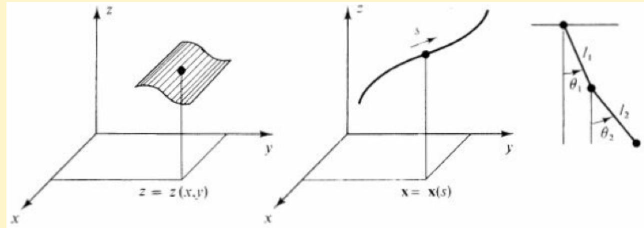
partial derivatives!
(all other variables are kept constant)

or, in a compact way:

$$dx_i = \sum_{\sigma=1}^{n-k} \frac{\partial x_i}{\partial q_\sigma} dq_\sigma + \frac{\partial x_i}{\partial t} dt \qquad i = 1, \ldots, n$$

$$x_1 = x_1(q_1, q_2, \ldots, q_{n-k}, t)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$x_n = x_n(q_1, q_2, \ldots, q_{n-k}, t)$$

$$dx_i = \sum_{\sigma=1}^{n-k} \frac{\partial x_i}{\partial q_\sigma} dq_\sigma + \frac{\partial x_i}{\partial t} dt \qquad i = 1, \ldots, n$$



Virtual displacements:

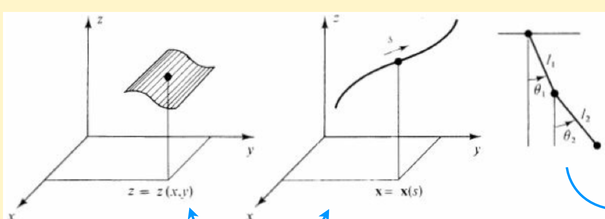
$$\delta x_i = \sum_{\sigma=1}^{n-k} \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma \qquad \text{i = 1, ..., n}$$

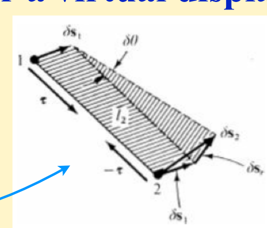
are defined as infinitesimal, instantaneous displacements of the coordinates consistent with the constraints.

D'Alembert's Principle

based on FW-14

The forces of constraint do no work under a virtual displacement:





rotation about point 1 (perpendicular to tension)

forces of constraint are perpendicular to the direction of motion and thus they do no work

$$\delta W = \boldsymbol{\tau} \cdot \delta \mathbf{s}_1 - \boldsymbol{\tau} \cdot \delta \mathbf{s}_2 = -\boldsymbol{\tau} \cdot \delta \mathbf{s}_r = 0$$

We can rewrite Newton's 2nd law as:

$$\dot{p}_i = F_i^{(a)} + R_i \qquad i = 1, \ldots, n \qquad \textbf{or} \qquad \sum_i (F_i^{(a)} + R_i - \dot{p}_i)\, \delta x_i = 0$$

applied force — **reaction force (constraint)**

$$\sum_i R_i\, \delta x_i = 0$$

D'Alembert's principle:

$$\sum_i (F_i^{(a)} - \dot{p}_i)\, \delta x_i = 0$$

forces of constraint have disappeared!

Lagrange's equations

based on FW-15

We want to rewrite D'Alembert's principle in terms of the generalized coordinates.

$$\sum_i (F_i^{(a)} - \dot{p}_i)\,\delta x_i = 0$$

The applied force piece:

$$\delta W = \sum_i F_i\,\delta x_i = \sum_{\sigma=1}^{n-k}\left(\sum_{i=1}^{n} F_i \frac{\partial x_i}{\partial q_\sigma}\right)\delta q_\sigma = \sum_{\sigma=1}^{n-k} Q_\sigma\,\delta q_\sigma$$

The virtual work done by applied forces under virtual displacement

$$\delta x_i = \sum_{\sigma=1}^{n-k} \frac{\partial x_i}{\partial q_\sigma}\,\delta q_\sigma$$

$$Q_\sigma \equiv \sum_{i=1}^{n} F_i \frac{\partial x_i}{\partial q_\sigma}$$

generalized forces

can be calculated directly from this definition, or from computing the virtual work done by applied forces for virtual displacement along a given generalized coordinate

Juggling with time derivatives:

$$dx_i = \sum_{\sigma=1}^{n-k} \frac{\partial x_i}{\partial q_\sigma}\,dq_\sigma + \frac{\partial x_i}{\partial t}\,dt \qquad i = 1, \ldots, n$$

$$x_i = x_i(q_1, q_2, \ldots, q_{n-k}, t)$$

$$\frac{dx_i}{dt} \equiv \dot{x}_i = \sum_\sigma \left(\frac{\partial x_i}{\partial q_\sigma}\right)\dot{q}_\sigma + \left(\frac{\partial x_i}{\partial t}\right) \qquad i = 1, \ldots, n$$

only functions of q and t

$$\dot{x}_i = \dot{x}_i(q_1, \ldots, q_{n-k}; \dot{q}_1, \ldots, \dot{q}_{n-k}; t) \qquad i = 1, \ldots, n$$

function of generalized coordinates, generalized velocities and time - all physically independent variables!
(if specified at a given time, subsequent motion of the system is determined)

useful formulas:

$$\frac{\partial \dot{x}_i}{\partial \dot{q}_\sigma} = \frac{\partial x_i}{\partial q_\sigma}$$

other variables are kept constant when taking partial derivatives

$$\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_\sigma}\right) = \frac{\partial}{\partial q_\sigma}\left(\frac{dx_i}{dt}\right)$$

$$\text{lhs} = \sum_\lambda \left(\frac{\partial}{\partial q_\lambda}\frac{\partial x_i}{\partial q_\sigma}\right)\dot{q}_\lambda + \frac{\partial}{\partial t}\frac{\partial x_i}{\partial q_\sigma}$$

$$\text{rhs} = \sum_\lambda \left(\frac{\partial}{\partial q_\sigma}\frac{\partial x_i}{\partial q_\lambda}\right)\dot{q}_\lambda + \frac{\partial}{\partial q_\sigma}\frac{\partial x_i}{\partial t}$$

the order of partial derivatives can be interchanged

$$\sum_i \left(F_i^{(a)} - \dot{p}_i\right) \delta x_i = 0$$

The momentum piece:

$$\sum_i \dot{p}_i\, \delta x_i = \sum_i m_i \ddot{x}_i\, \delta x_i = \sum_\sigma \left(\sum_i m_i \ddot{x}_i \frac{\partial x_i}{\partial q_\sigma} \right) \delta q_\sigma$$

$$\dot{p}_i = m_i \ddot{x}_i$$

$$\delta x_i = \sum_{\sigma=1}^{n-k} \frac{\partial x_i}{\partial q_\sigma} \delta q_\sigma$$

$$\sum_i m_i \ddot{x}_i \frac{\partial x_i}{\partial q_\sigma} \equiv \sum_i m_i \frac{d\dot{x}_i}{dt} \frac{\partial x_i}{\partial q_\sigma} = \sum_i m_i \left[\frac{d}{dt}\left(\dot{x}_i \frac{\partial x_i}{\partial q_\sigma} \right) - \dot{x}_i \frac{d}{dt} \frac{\partial x_i}{\partial q_\sigma} \right]$$

$$\frac{\partial \dot{x}_i}{\partial \dot{q}_\sigma} = \frac{\partial x_i}{\partial q_\sigma}$$

$$\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_\sigma} \right) = \frac{\partial}{\partial q_\sigma}\left(\frac{dx_i}{dt} \right)$$

$$\sum_i m_i \frac{d}{dt}\left(\dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_\sigma} \right) = \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_\sigma}\left(\frac{1}{2} \sum_i m_i \dot{x}_i^2 \right) \right]$$

$$\sum_i m_i \dot{x}_i \frac{\partial}{\partial q_\sigma} \frac{dx_i}{dt} = \frac{\partial}{\partial q_\sigma}\left(\frac{1}{2} \sum_i m_i \dot{x}_i^2 \right)$$

$$\sum_i \dot{p}_i\, \delta x_i = \sum_\sigma \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} \right) \delta q_\sigma$$

kinetic energy:

$$T = \frac{1}{2} \sum_i m_i \dot{x}_i^2 = T(q_1, \ldots, q_{n-k}; \dot{q}_1, \ldots, \dot{q}_{n-k}; t)$$

Now we can rewrite D'Alembert's principle in terms of the generalized coordinates:

$$\sum_i \left(F_i^{(a)} - \dot{p}_i\right) \delta x_i = 0$$

$$\delta W = \sum_i F_i\, \delta x_i = \sum_{\sigma=1}^{n-k} \left(\sum_{i=1}^{n} F_i \frac{\partial x_i}{\partial q_\sigma} \right) \delta q_\sigma = \sum_{\sigma=1}^{n-k} Q_\sigma\, \delta q_\sigma$$

generalized forces

$$\sum_i \dot{p}_i\, \delta x_i = \sum_\sigma \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} \right) \delta q_\sigma$$

$$T = \frac{1}{2} \sum_i m_i \dot{x}_i^2 = T(q_1, \ldots, q_{n-k}; \dot{q}_1, \ldots, \dot{q}_{n-k}; t)$$

kinetic energy

$$\sum_\sigma \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} - Q_\sigma \right) \delta q_\sigma = 0$$

all q are independent and arbitrary

Lagrange's equations:

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} = Q_\sigma \qquad \sigma = 1, \ldots, n-k$$

n-k equations for n-k independent generalized coordinates
(equivalent to Newton's 2nd law)

For conservative forces (potential energy depends only on the position):

$$V(x_1, \ldots, x_n) = V(q_1, \ldots, q_{n-k}, t)$$

$$\frac{\partial V}{\partial \dot{q}_\sigma} = 0$$

generalized forces are given by negative gradients with respect to corresponding generalized coordinate:

$$Q_\sigma \equiv \sum_i F_i \frac{\partial x_i}{\partial q_\sigma} = -\sum_i \left[\frac{\partial}{\partial x_i} V(x_1, \ldots, x_n)\right] \frac{\partial x_i}{\partial q_\sigma} = -\frac{\partial}{\partial q_\sigma} V(q_1, \ldots, q_{n-k}, t)$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} = Q_\sigma \qquad \sigma = 1, \ldots, n-k$$

Lagrange's equations

for conservative forces:

$$L \equiv T - V$$

Lagrangian

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0 \qquad \sigma = 1, \ldots, n-k$$