Using Lagrange’s equations

**Pendulum:**

\[ T = \frac{1}{2}m(l\dot{\theta})^2 \]

\[ V = -mg \cos \theta + \text{const} \]

\[ L \equiv T - V \]

\[ L = \frac{1}{2}ml^2 \ddot{\theta}^2 + mgl \cos \theta - \text{const} \]

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0 \quad \sigma = 1, \ldots, n - k \]

\[ \ddot{\theta} = -\frac{g}{l} \sin \theta \]

$\theta$ is the generalized coordinate

**pendulum equation, small small-amplitude approximation - oscillations with**

\[ \omega = (g/l)^{1/2} \]
Bead on a Rotating Wire Hoop: hoop rotates with constant angular velocity about an axis perpendicular to the plane of the hoop and passing through the edge of the hoop. No friction, no gravity.

$\theta$ is the generalized coordinate

$x = a \cos \omega t + a \cos (\omega t + \theta)$

$y = a \sin \omega t + a \sin (\omega t + \theta)$

$\dot{x} = -a\omega \sin \omega t - a(\omega + \dot{\theta}) \sin (\omega t + \theta)$

$\dot{y} = a\omega \cos \omega t + a(\omega + \dot{\theta}) \cos (\omega t + \theta)$

$L = T - V$

$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

$\cos \omega t \cos (\omega t + \theta) + \sin \omega t \sin (\omega t + \theta) = \cos (\omega t + \theta - \omega t) = \cos \theta$

$T = L = \frac{1}{2}ma^2[\omega^2 + (\omega + \dot{\theta})^2 + 2\omega(\omega + \dot{\theta}) \cos \theta]$

$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0 \quad \sigma = 1, \ldots, n - k$

$\ddot{\theta} + \omega^2 \sin \theta = 0$

pendulum equation
Moving inside a cone: A particle of mass $m$ is constrained to move on the inside surface of a smooth cone of half-angle $\alpha$. The particle is subject to a gravitational force. Determine a set of generalized coordinates and find Lagrange's equations of motion.
Rotating pendulum: The point of support of a simple pendulum of length $b$ moves on a massless rim of radius $a$ rotating with constant angular velocity $\omega$. Find the equation of motion for the angle $\theta$. 
Pendulum with horizontally moving support:

A simple pendulum of mass $m_2$ with a mass $m_1$ at the point of support which can move on a horizontal line in the plane in which $m_2$ moves. Find the equation of motion for $x$ and the angle $\phi$. 
Motivation: We will be able to obtain the whole set of Lagrange’s equations from a single variational principle.

Find the function $y(x)$ that makes the functional

$$\phi \equiv \phi[y(x), y'(x), x]$$

an extremum (for us minimum).

Problem:

Find the function $y(x)$ that makes

$$I \equiv \int_{x_1}^{x_2} \phi(y, y', x) \, dx$$

an extremum (for us minimum).
Examples:

What function \( y(x) \) minimizes the distance between 1 and 2?

\[
ds = [(dx)^2 + (dy)^2]^{1/2} = [1 + (y')^2]^{1/2} \, dx
\]

the functional for this problem is:

\[
\phi = [1 + (y')^2]^{1/2}
\]

What shape of the wire minimizes the time of travel from point 1 to 2?

(no friction, uniform gravitational field)

\[
t_{12} = \int_{1}^{2} \frac{ds}{v}
\]

\[\frac{1}{2}mv^2 = mgy\]

the functional for this problem is:

\[
\phi = \left[ \frac{1 + (y')^2}{2gy} \right]^{1/2}
\]

the solution is called a brachistochrone
Problem: Find the function $y(x)$ that makes $I$ an extremum.

Solution:

Let $y(x)$ be the solution, and construct infinitesimal arbitrary functions, satisfying:

$\eta(x_1) = \eta(x_2) = 0$

Let’s calculate the integral for $Y(x)$:

$$I(\epsilon) = \int_{x_1}^{x_2} \phi[Y(x), Y'(x), x] \, dx = \int_{x_1}^{x_2} \phi[y(x) + \epsilon \eta(x), y'(x) + \epsilon \eta'(x), x] \, dx$$

Taylor series expansion about $\epsilon = 0$

$$(which \ is \ the \ solution \ for \ \epsilon \to 0)$$

$I(\epsilon)$ has an extremum for $\epsilon = 0!$
Problem: Find the function $y(x)$ that makes an extremum.

Solution (continued):

\[ \int_{x_1}^{x_2} \left[ \frac{\partial \phi}{\partial y} \eta(x) + \frac{\partial \phi}{\partial y'} \frac{d}{dx} \eta(x) \right] \, dx = 0 \]

\[ \frac{\partial \phi}{\partial y'} \frac{d}{dx} \eta(x) = \frac{d}{dx} \left[ \frac{\partial \phi}{\partial y} \eta(x) \right] - \eta(x) \frac{d}{dx} \frac{\partial \phi}{\partial y'} \]

\[ \int_{x_1}^{x_2} \frac{d}{dx} \left[ \frac{\partial \phi}{\partial y'} \eta(x) \right] \, dx = \left[ \eta(x) \frac{\partial \phi}{\partial y'} \right]_{x_1}^{x_2} = 0 \]

\[ \eta(x_1) = \eta(x_2) = 0 \]

Euler-Lagrange equation for the variational problem!
Example:

What function \( y(x) \) minimizes the distance between 1 and 2?

The functional for this problem is:

\[
\phi = \left[ 1 + (y')^2 \right]^{1/2}
\]

and the solution is obtained from Euler-Lagrange equation:

\[
\frac{d}{dx} \frac{y'}{\left[ 1 + (y')^2 \right]^{1/2}} = 0
\]

\[
y'' = 0
\]

\[
y(x) = ax + b
\]

straight line (as expected)
Connection of what we just did with variations:

\[
Y(x) = y(x) + \epsilon \eta(x)
\]

infinitesimal

arbitrary functions, satisfying:

\[
\eta(x_1) = \eta(x_2) = 0
\]

Taylor series expansion about \( \epsilon = 0 \)

\[
I(\epsilon) = \int_{x_1}^{x_2} \phi[Y(x), Y'(x), x] \, dx = \int_{x_1}^{x_2} \phi[y(x) + \epsilon \eta(x), y'(x) + \epsilon \eta'(x), x] \, dx
\]

variation of the functional:

\[
\phi[Y(x), Y'(x), x] - \phi[y(x), y'(x), x] \equiv \delta \phi
\]

Taylor series expansion:

\[
\delta \phi = \frac{\partial \phi}{\partial y} \delta y + \frac{\partial \phi}{\partial y'} \delta y'
\]

then:

\[
Y(x) - y(x) = \epsilon \eta(x) \equiv \delta y(x)
\]

\[
\delta y(x_1) = \delta y(x_2) = 0
\]

\[
Y'(x) - y'(x) = \epsilon \eta'(x) \equiv \delta y'(x)
\]

\[
\delta y'(x) = \frac{d}{dx} \delta y(x)
\]
Connection of what we just did with variations:

\[ I(\epsilon) = \int_{x_1}^{x_2} \phi(y, y', x) + \epsilon \int_{x_1}^{x_2} \left[ \frac{\partial \phi}{\partial y} \eta(x) + \frac{\partial \phi}{\partial y'} \eta'(x) \right] \, dx + O(\epsilon^2) \]

\[ \left. \frac{dI(\epsilon)}{d\epsilon} \right|_{\epsilon=0} = 0 \]

\[ \int_{x_1}^{x_2} \left[ \frac{\partial \phi}{\partial y} \eta(x) + \frac{\partial \phi}{\partial y'} \eta'(x) \right] \, dx = 0 \]

\[ \delta \phi = \frac{\partial \phi}{\partial y} \delta y + \frac{\partial \phi}{\partial y'} \delta y' \]

\[ \delta I = \int_{x_1}^{x_2} \delta \phi \, dx = \int_{x_1}^{x_2} \left( \frac{\partial \phi}{\partial y} \delta y + \frac{\partial \phi}{\partial y'} \delta y' \right) \, dx \]

\[ = \int_{x_1}^{x_2} \delta y \left( \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial y'} \frac{d}{dx} \frac{d}{dy'} \right) \, dx \]

\[ \delta y(x_1) = \delta y(x_2) = 0 \]

\[ \delta I = 0 \]

\[ \frac{d}{dx} \frac{\partial \phi}{\partial y'} - \frac{\partial \phi}{\partial y} = 0 \]

Euler-Lagrange equation for the variational problem!