Hamilton’s principle

Variational statement of mechanics:
(for conservative forces)

Equivalent to Newton’s laws!

Based on FW-18

the particle takes the path
that minimizes the integrated
difference of the kinetic and
potential energies
Generalization to a system with \( n \) degrees of freedom:

\[
0 = \delta \int_{t_1}^{t_2} L(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t) \, dt \\
= \int_{t_1}^{t_2} \sum_{\sigma=1}^{n} \left( \frac{\partial L}{\partial \dot{q}_\sigma} \delta q_\sigma + \frac{\partial L}{\partial q_\sigma} \delta q_\sigma \right) \, dt \\
\delta q_\sigma(t_1) = \delta q_\sigma(t_2) = 0 \quad \sigma = 1, \ldots, n
\]

\[
\delta q_\sigma(t) = \epsilon \eta_\sigma(t) \quad \sigma = 1, \ldots, n
\]

\[
\delta \dot{q}_\sigma = \frac{d}{dt} \delta q_\sigma
\]

if all the generalized coordinates are independent

for \( k \) holonomic constraints:

\[
f_j(q_1, \ldots, q_n, t) = c_j \quad j = 1, \ldots, k
\]

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0 \quad \sigma = 1, \ldots, n
\]

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0 \quad \sigma = 1, \ldots, n - k.
\]
Forces of constraint

Often it is useful to incorporate (some) constraints into Hamilton’s principle:

\[ \int_{t_1}^{t_2} \sum_{\sigma} \delta q_\sigma \left( \frac{\partial L}{\partial q_\sigma} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} \right) dt = 0 \]

\[ \delta f_j = \sum_{\sigma=1}^{n} \frac{\partial f_j}{\partial q_\sigma} \delta q_\sigma = 0 \quad j = 1, \ldots, k \]

Lagrange multipliers (can be chosen so that coefficients of \( k \) dependent variations of coordinates vanish.

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q_\sigma} \quad \sigma = 1, \ldots, n \]

\[ f_j(q_1, \ldots, q_n, t) = c_j \quad j = 1, \ldots, k \]

\( n-k \) independent coordinates, \( k \) constraints

Adding 0

\( n+k \) equations for \( n+k \) unknowns
Lagrange multipliers determine reaction forces:

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q_\sigma}, \quad \sigma = 1, \ldots, n
\]

\[
f_j(q_1, \ldots, q_n, t) = c_j, \quad j = 1, \ldots, k
\]

\[
L = T - V
\]

Lagrange’s equations:

\[
\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} = -\frac{\partial V}{\partial q_\sigma} + \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q_\sigma}, \quad \sigma = 1, \ldots, n
\]

forces of constraint (reaction forces) (given by Lagrange multipliers)
we can choose to include any one or all constraints, solve n+k equations for n+k unknowns, including Lagrange multipliers, and determine reaction forces of interest.

(reaction forces correspond to variations of generalized coordinates that violate the constraints)
Using Lagrange’s equations

**Pendulum:**

\[ T = \frac{1}{2}m(l\dot{\theta})^2 \]

\[ V = -mgl\cos \theta + \text{const} \]

\[ L \equiv T - V \]

\[ L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos \theta - \text{const} \]

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0 \quad \sigma = 1, \ldots, n-k \]

\[ \frac{d}{dt} ml^2 \dot{\theta} - (-mgl \sin \theta) = 0 \]

\[ \ddot{\theta} = -\frac{g}{l} \sin \theta \]

\( \theta \) is the generalized coordinate

Based on FW-16

**Pendulum equation, small small-amplitude approximation - oscillations with**

\[ \omega = (g/l)^{1/2} \]
Pendulum: (with \( r \) and \( \theta \) as generalized coordinates)

\[
L = \frac{1}{2}m(r^2 + r^2\dot{\theta}^2) +mgr \cos \theta + \text{const}
\]

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = \sum_{j=1}^{n} \lambda_j \frac{\partial f_j}{\partial q_\sigma}, \quad \sigma = 1, \ldots, n
\]

\[
m(r - r\dot{\theta}^2) - mg \cos \theta = \lambda \equiv Q_r
\]

\[
\frac{d}{dt} mr^2\dot{\theta} + mgr \sin \theta = 0
\]

\[
-\lambda = \tau = ml\dot{\theta}^2 + mg \cos \theta
\]

\[
\dot{\theta} = -\frac{g}{l} \sin \theta
\]

3 eqns. for 3 unknown

\[
r = l \quad \dot{r} = \ddot{r} = 0
\]

The force of constraint is the tension:

\[
\delta W = Q_r \delta r = -\tau \delta r
\]

\[
Q_r = \lambda = -\tau
\]

(reaction forces correspond to variations of generalized coordinates that violate the constraints)

Tension force given by the the centrifugal force and the \( r \)-component of the gravitational force
Atwood’s machine (elevator):

\[ L = \frac{1}{2}m_1 l_1^2 + \frac{1}{2}m_2 l_2^2 + m_1 gl_1 + m_2 gl_2 + \text{const} \]

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q_\sigma} \quad \sigma = 1, \ldots, n
\]

\[ l_1 + l_2 = l \]

\[ \ddot{l}_1 = -\ddot{l}_2 \]

\[ \ddot{l}_1 = \frac{m_1 - m_2}{m_1 + m_2} g \]

\[ Q_1 = Q_2 = \lambda = -\frac{2m_1 m_2}{m_1 + m_2} g \]

Constraint:

\[ \lambda (\delta l_1 + \delta l_2) = 0 \]

3 eqns. for 3 unknowns

The forces of constraint are tensions:

\[ \delta W = Q_1 \delta l_1 + Q_2 \delta l_2 \]

\[ \delta W = Q_1 \delta l_1 = -\tau_1 \delta l_1 \]

\[ -\lambda = \tau_1 = \tau_2 \]

(Reaction forces correspond to variations of generalized coordinates that violate the constraints)