Hamilton’s principle based on FW-18

Variational statement of mechanics: (for conservative forces)

\[ L = T - V \]

\[ \delta \int_{t_1}^{t_2} L[q(t), \dot{q}(t), t] \, dt = 0 \]

\[ \delta q(t_1) = \delta q(t_2) = 0 \]

\[ d \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0 \]

Equivalent to Newton’s laws!

Generalization to a system with \( n \) degrees of freedom:

\[ 0 = \delta \int_{t_1}^{t_2} L[q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n; t] \, dt \]

\[ = \int_{t_1}^{t_2} \sum_{\sigma=1}^{n} \left( \frac{\partial L}{\partial q_\sigma} \delta q_\sigma + \frac{\partial L}{\partial \dot{q}_\sigma} \delta \dot{q}_\sigma \right) \, dt \]

\[ \delta \dot{q}_\sigma(t_1) = \delta \dot{q}_\sigma(t_2) = 0 \quad \sigma = 1, \ldots, n \]

\[ \delta q_\sigma(t) = \epsilon \eta_\sigma(t) \quad \sigma = 1, \ldots, n \]

\[ \delta \dot{q}_\sigma = \frac{d}{dt} \delta q_\sigma \]

\[ \int_{t_1}^{t_2} \sum_{\sigma=1}^{n} \delta q_\sigma \left( \frac{\partial L}{\partial q_\sigma} + \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} \right) \, dt = 0 \]

if all the generalized coordinates are independent

\[ d \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0 \quad \sigma = 1, \ldots, n \]

for \( k \) holonomic constraints:

\[ f_j(q_1, \ldots, q_n, t) = c_j \quad j = 1, \ldots, k \]

\[ d \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0 \quad \sigma = 1, \ldots, n-k \]
Forces of constraint

Often it is useful to incorporate (some) constraints into Hamilton’s principle:

\[
\int_{t_1}^{t_2} \left[ \sum_{\sigma} \delta q^{\sigma} \left( \frac{\partial L}{\partial q^{\sigma}} - \frac{d}{dt} \frac{\partial L}{\partial q^{\sigma}_{\dot{}}} \right) \right] dt = 0
\]

\[
f_j(q_1, \ldots, q_n, t) = c_j \quad j = 1, \ldots, k
\]

\[
\delta f_j = \sum_{\sigma=1}^{n} \frac{\partial f_j}{\partial q^{\sigma}_{\dot{}}} \delta q^{\sigma}_{\dot{}} = 0 \quad j = 1, \ldots, k
\]

\[n-k\] independent coordinates, 
\[k\] constraints

Adding 0

Lagrange multipliers (can be chosen so that coefficients of \(k\) dependent variations of coordinates vanish.

\[
\frac{d}{dt} \frac{\partial L}{\partial q^{\sigma}_{\dot{}}} - \frac{\partial L}{\partial q^{\sigma}} = \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q^{\sigma}} \quad \sigma = 1, \ldots, n
\]

\[
f_j(q_1, \ldots, q_n, t) = c_j \quad j = 1, \ldots, k
\]

\[n+k\] equations for \(n+k\) unknowns

Lagrange multipliers determine reaction forces:

\[
\frac{d}{dt} \frac{\partial L}{\partial q^{\sigma}_{\dot{}}} - \frac{\partial L}{\partial q^{\sigma}} = \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q^{\sigma}}
\]

\[
f_j(q_1, \ldots, q_n, t) = c_j \quad j = 1, \ldots, k
\]

\[L \equiv T - V\]

\[Q^\sigma = \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q^{\sigma}}\]

\[
Q^\sigma = -\frac{\partial V}{\partial q^{\sigma}} + Q^\sigma
\]

Lagrange’s equations:

\[
\frac{d}{dt} \frac{\partial T}{\partial q^{\sigma}_{\dot{}}} - \frac{\partial T}{\partial q^{\sigma}} = Q^\sigma \quad \sigma = 1, \ldots, n
\]

\[
\delta W = \sum_{\sigma} Q^\sigma \delta q^{\sigma}
\]

forces of constraint (reaction forces) (given by Lagrange multipliers)

we can choose to include any one or all constraints, solve \(n+k\) equations for \(n+k\) unknowns, including Lagrange multipliers, and determine reaction forces of interest.

(reaction forces correspond to variations of generalized coordinates that violate the constraints)
Using Lagrange’s equations

**Pendulum:**

\[ T = \frac{1}{2} m (l \dot{\theta})^2 \]

\[ V = -mg l \cos \theta + \text{const} \]

\[ L = T - V \]

\[ L = \frac{1}{2} ml^2 \dot{\theta}^2 + mg l \cos \theta - \text{const} \]

\[ \frac{d}{dt} \left( ml^2 \dot{\theta} - (-mg \sin \theta) \right) = 0 \]

\[ \ddot{\theta} = -\frac{g}{l} \sin \theta \]

\[ \theta \] is the generalized coordinate

**Pendulum:**

\[ L = \frac{1}{2} m (r^2 + r^2 \dot{\theta}^2) + mgr \cos \theta + \text{const} \]

constraint:

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_f} - \frac{\partial L}{\partial q_f} = \sum_{i=1}^{n} \lambda_i \frac{\partial \phi_i}{\partial q_f}, \quad \sigma = 1, \ldots, n \]

\[ r = l \]

\[ \delta r = 0 \]

\[ m(r - r \dot{\theta}^2) - mg \cos \theta = \lambda = Q, \]

\[ \frac{d}{dt} mr^2 \dot{\theta} + mgr \sin \theta = 0 \]

\[ 3 \text{ eqns. for 3 unknowns} \]

\[ r = l \]

\[ \dot{r} = \ddot{r} = 0 \]

\[ -\lambda = \tau = m \dot{\theta}^2 + mg \cos \theta \]

\[ \ddot{\theta} = -\frac{g}{l} \sin \theta \]

**Pendulum equation, small small-amplitude approximation - oscillations with**

\[ \omega = \left(\frac{g}{l}\right)^{1/2} \]

**Pendulum:**

\[ (\text{with } r \text{ and } \theta \text{ as generalized coordinates}) \]

\[ \text{the force of constraint is the tension:} \]

\[ \delta W = Q_r \delta r = -\tau \delta r \]

\[ Q_r = \lambda = -\tau \]

(reaction forces correspond to variations of generalized coordinates that violate the constraints)

\[ \text{tension force given by the centrifugal force and the r-component of the gravitational force} \]
Atwood’s machine (elevator):

\[ L = {\frac{1}{2}}m_1 \dot{l}_1^2 + {\frac{1}{2}}m_2 \dot{l}_2^2 + m_1 gl_1 + m_2 gl_2 + \text{const} \]

Constraint:

\[ \sum \frac{\partial L}{\partial \dot{q}_\sigma} \frac{\partial}{\partial q_\sigma} = \sum \lambda_i \frac{\partial f_i}{\partial q_\sigma}, \quad \sigma = 1, \ldots, n \]

\[ \dot{l}_1 + \dot{l}_2 = l \]

\[ \dot{\lambda}(\delta l_1 + \delta l_2) = 0 \]

The forces of constraint are tensions:

\[ \delta W = Q_1 \delta l_1 + Q_2 \delta l_2 \]

\[ \delta W = Q_1 \delta l_1 = -\tau_1 \delta l_1 \]

\[ -\lambda = \tau_1 = \tau_2 \]

(reaction forces correspond to variations of generalized coordinates that violate the constraints)

3 eqns. for 3 unknown:

\[ \dot{l}_1 = -\dot{l}_2 \]

\[ \dot{l}_1 = \frac{m_1 - m_2}{m_1 + m_2} g \]

\[ Q_1 = Q_2 = \lambda = -\frac{2m_1 m_2}{m_1 + m_2} g \]