Consider a system on $N$ particles with all their relative separations fixed: it has 3 translational and 3 rotational degrees of freedom.

Motion with One Arbitrary Fixed Point:

\[
\frac{d\mathbf{r}}{dt}_{\text{inertial}} = \frac{d\mathbf{r}}{dt}_{\text{body}} + \omega \times \mathbf{r}
\]

\[
\frac{d\mathbf{r}}{dt}_{\text{body}} = \sum_{i=1}^{3} \dot{\mathbf{e}}_i \frac{dx_i}{dt}
\]

\[
\left( \frac{d\mathbf{r}}{dt} \right)_{\text{inertial}} = \omega \times \mathbf{r}
\]
Kinetic Energy:

\[
\frac{d\mathbf{r}}{dt}_{\text{inertial}} = \mathbf{\omega} \times \mathbf{r}
\]

\[
|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2.
\]

continuum limit:

\[
T = \frac{1}{2} \int d^3r \rho(r) [\mathbf{\omega}^2 r^2 - (\mathbf{\omega} \cdot \mathbf{r})^2]
\]

It is convenient to define the **inertia tensor**:

\[
I_{ij} \equiv \sum_p m_p (\delta_{ij} r_p^2 - x_{pi} x_{pj})
\]

\[
I_{ij} \equiv \int d^3r \rho(r) (\delta_{ij} r^2 - x_i x_j)
\]

In a body-fixed frame it is constant and depends only on the mass distribution.

The kinetic energy has a simple form:

\[
T = \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} I_{ij} \omega_i \omega_j
\]

Both \( I \) and \( \mathbf{\omega} \) depend on the choice of the coordinate system, but \( T \) does not!
Angular momentum:

\[
\frac{d\mathbf{r}}{dt}_{\text{inertial}} = \mathbf{\omega} \times \mathbf{r}
\]

\[
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})
\]

continuum limit:

\[
\mathbf{L} = \int d^3r \, \rho(r)[r^2\mathbf{\omega} - (\mathbf{\omega} \cdot \mathbf{r})\mathbf{r}]
\]

using the inertia tensor it can be written as:

\[
L_i = \sum_{j=1}^{3} I_{ij}\omega_j
\]

\[
I_{ij} = \int d^3r \, \rho(r)(\delta_{ij}r^2 - x_ix_j)
\]

\[
T = \frac{1}{2} \sum_{i=1}^{3} L_i \omega_i = \frac{1}{2} \mathbf{L} \cdot \mathbf{\omega}
\]

I, \(\mathbf{\omega}\) and \(\mathbf{L}\) depend on the choice of the coordinate system!

and it is related to the kinetic energy:
General motion with No Fixed Point:

- Body-fixed frame with the origin at the center of mass
- A parallel copy of the inertial frame with the same origin as the body fixed frame - the center-of-mass frame (in general it is not an inertial frame)

\[ \mathbf{r} = \mathbf{R} + \mathbf{r}' \]

\[ \frac{d}{dt} \left( \frac{d\mathbf{r}}{dt} \right)_{\text{inertial}} = \frac{d\mathbf{R}}{dt} + \frac{d\mathbf{r}'}{dt}_{\text{inertial}} \]

\[ \frac{d\mathbf{r}'}{dt}_{\text{cm}} = \frac{d\mathbf{r}'}{dt}_{\text{body}} + \mathbf{\omega} \times \mathbf{r}' \]

0 for rigid bodies
Kinetic Energy and Angular Momentum:

Body-fixed frame with the origin at the center of mass.

A parallel copy of the inertial frame with the same origin as the body fixed frame - the center-of-mass frame.

(in general it is not an inertial frame)

Review:

\[ M\ddot{\mathbf{R}} = \sum_p F_p^{(e)} = F^{(e)} \]

\[ T = \frac{1}{2}M\dot{\mathbf{R}}^2 + T' \]

\[ \mathbf{L} = \mathbf{R} \times (M\dot{\mathbf{R}}) + \mathbf{L}' \]

Formulas for \( T' \) and \( \mathbf{L}' \) are identical to the case of a motion with one fixed point!
Inertia tensor

We have to learn how to evaluate the inertia tensor in a body-fixed frame. First, let’s define the inertia tensor in a body-fixed frame with the origin at the center of mass:

\[
\bar{I}_{ij} = \sum_p m_p (\delta_{ij} x_p^2 - x_{pi} x_{pj})
\]

\[
\bar{I}_{ij} = \int d^3 \rho(x) (\delta_{ij} x^2 - x_i x_j)
\]

Then the inertia tensor in a body-fixed frame with the origin displaced by \(\mathbf{a}\) is given by:

\[
I_{ij} = \sum_p m_p (\delta_{ij} y_p^2 - y_{pi} y_{pj})
\]

\[
\sum_m m_i r_i = 0 \quad \sum_m m_i v_i = 0
\]

center-of-mass relations

\[
I_{ij} = \bar{I}_{ij} + M (a^2 \delta_{ij} - a_i a_j)
\]

parallel-axis theorem!
Example (a uniform disk of radius $a$):

- **Moment of inertia about a perpendicular axis through the center of the disk**
  - (easy to calculate in cylindrical polar coordinates)

- **Moment of inertia about a perpendicular axis through the edge of the disk**
  - (would be tedious to calculate directly)

\[ I_{ij} = \bar{I}_{ij} + M(a^2 \delta_{ij} - a_i a_j) \]

\[ I_{33} = \bar{I}_{33} + Ma^2 \]
**Principal axes** (a coordinate system in which the inertia tensor is diagonal):

\[ I_{ij} = I_i \delta_{ij} \]

all the formulas simplify in such a coordinate system

Diagonalizing a real symmetric matrix:

\[ \sum_{j=1}^{3} I_{ij} e_j = \lambda e_i \]
\[ \sum_{j=1}^{3} (I_{ij} - \lambda \delta_{ij}) e_j = 0 \]

similar to solving for normal modes, \( v \rightarrow I \) and \( m \rightarrow 1 \)

non-trivial solution only if:

\[ \det | I_{ij} - \lambda \delta_{ij} | = 0 \]

we get 3 eigenvalues, 3 eigenvectors, and we can form the modal matrix:

\[ \mathcal{A} = \begin{bmatrix} e^{(1)} & e^{(2)} & e^{(3)} \end{bmatrix} \]
\[ \mathcal{A}^T \mathcal{A} = \mathcal{A}^{-1} \mathcal{A} = 1 \]

new orthonormal basis - principal axes
Modal matrix diagonalizes the inertia tensor:

\[ \mathbf{A}^{-1} \mathbf{A} = \mathbf{A}^T \mathbf{A} = \lambda \mathbf{I} = \begin{bmatrix} I_1 & I_2 & I_3 \end{bmatrix} \]

we can define a new set of angular velocities:

\[ \mathbf{\omega} = \mathbf{A} \xi \]

\[ \xi = \mathbf{A}^{-1} \mathbf{\omega} = \mathbf{A}^T \mathbf{\omega} \]

projections of \( \mathbf{\omega} \) along the principal axes

Kinetic Energy becomes:

\[ T = \frac{1}{2} \mathbf{\omega}^T \mathbf{I} \mathbf{\omega} = \frac{1}{2} (\xi^T \mathbf{A}^T \mathbf{I} \mathbf{A} \xi) = \frac{1}{2} \xi^T \lambda \xi = \frac{1}{2} \sum_{s=1}^{3} \xi^2_s I_s \]
Angular Momentum in the new coordinate system:

\[
L_{\text{new}} = \mathcal{A}^T L = \begin{bmatrix}
\hat{e}^{(1)} \cdot L \\
\hat{e}^{(2)} \cdot L \\
\hat{e}^{(3)} \cdot L
\end{bmatrix}
\]

Principal moments of inertia:

\[
I_s = (\mathcal{A}^T \mathcal{I} \mathcal{A})_{ss} = \int d^3 r \rho(r)[r^2 - (r \cdot \hat{e}^{(s)})^2]
\]

projections of \( r \) along principal axes
Euler’s Equations

Describing rotational motion:

![Euler's Equations Diagram]

valid in any inertial frame but also in the cm frame:

\[
\left(\frac{dL}{dt}\right)_{\text{inertial}} = \left(\frac{dL}{dt}\right)_{\text{cm}} = \sum_p r_p \times \Gamma_p^{(e)} = \Gamma^{(e)}
\]

in terms of body-fixed observables:

\[ L_s = I_s \omega_s \]

projecting onto the principal body axes

\[
(dL_s/dt)_{\text{body}} = I_s \, d\omega_s/dt
\]

\[
\omega_2 L_3 - \omega_3 L_2 = \omega_2 \omega_3 (I_3 - I_2)
\]

Euler’s equations:

![Euler's Equations Formula]

simple formulas, but limited use, since both angular velocities and torques are evaluated in a time dependent body-fixed frame (principal axes);

used mostly for torque-free motion or partially constrained motion (examples follow)
Applications - Compound Pendulum

**a rigid body constrained to rotate about a fixed axis:**
stationary both in the inertial frame and in body-fixed frame

**we can choose the coordinate system so that:**

\[ R = l \hat{e}_1 \]

**one degree of freedom**

ignoring friction, the only torque comes from gravity:

\[ \Gamma^{(e)} = -Mgl \sin \phi \]

\[ \Gamma^{(e)} = \sum_p m_p r_p \times g = MR \times g \]

\[ L_3 = \sum_{j=1}^{3} I_{3j} \omega_j = I_{33} \phi \]

\[ L_1 = \sum_{i,j} I_{ij} \omega_j \]

\[ r_1^2 = x_1^2 + x_2^2 \]

\[ I = I_{33} = \int d^3 x \rho(x)(x^2 - x_3^2) = \int d^3 x \rho(x)(x_1^2 + x_2^2) = \int d^3 x \rho(x)r_1^2 \]
for small oscillations, the motion is simple harmonic:

\[
\ddot{\phi} = -\Omega^2 \phi
\]

\[
\Omega = \left(\frac{Mgl}{I}\right)^{1/2}
\]

Let’s rewrite it in terms of the moment of inertia about the axis going through the center of mass:

\[
I = I_{33} = \bar{I}_{33} + Ml^2
\]

radius of gyration

\[
\bar{I}_{33} \equiv Mk^2
\]

equivalent radial distance of a point mass M leading to the same moment of inertia

\[
I = M(k^2 + l^2) = Mk^2
\]

radius of gyration about the axis through Q

\[
\Omega_Q^2 = \frac{g}{k^2 + l^2} = \frac{g}{l_1}
\]

\[
l_1 \equiv l + \frac{k^2}{l}
\]

\[
l_1 > l
\]
The oscillations frequencies and periods about the axis through \( Q \) and \( P \) are identical:

\[
\Omega_p^2 = \frac{Mg\bar{k}^2/l}{M[(\bar{l}_1 - l)^2 + \bar{k}^2]} = \frac{g}{l + \bar{k}^2/l} = \Omega_Q^2
\]

is used to measure \( g \)!
Response of a baseball bat to a transverse force: (we neglect gravity)

reaction force at the point of support

motion of the center of mass (N-2nd):

\[ M \ddot{\mathbf{R}} \cdot \hat{e}_2 = f - f_r \]
\[ \mathbf{R} \cdot \hat{e}_2 = l \dot{\phi} \]

the only direction \( R \) can move due to constraints

\[ M l \ddot{\phi} = f - f_r \]

torque equation:

\[ \frac{dL_3}{dt} = I_3 \ddot{\phi} = \Gamma^{(e)}_3 = f l' \]

\[ I_3 \equiv I = M (\bar{k}^2 + l^2) \]

\[ l \dot{\phi} \equiv M (l^2 + \bar{k}^2) \ddot{\phi} = f l' \]

if the force is applied at the center of percussion
the reaction force at the point of support vanishes!

if applied beyond that point, the reaction force has the same direction as applied force
Rolling and Sliding Billiard Ball

A ball is struck at the center \( h=0 \) with a horizontal force. When does it start rolling?

**Initial conditions:**
\[
 x = 0, \quad \dot{x} = v_0, \quad \phi = 0, \quad \dot{\phi} = 0
\]

**Force equation:**
\[
 F_f = -\mu g \dot{\phi}
\]

**Solution:**
\[
 \dot{x} = v_0 - \mu g t
\]

**Torque equation:**
\[
 \frac{dL_3}{dt} = I_3 \ddot{\phi} = \Gamma^{(e)} = F_f a
\]

\[
 I_3 = \frac{2}{5} M a^2
\]

\[
 a \ddot{\phi} = \frac{5}{2} \mu g t
\]

**Pure rolling without sliding occurs only for:**
\[
 \ddot{\phi} = \frac{5}{2} \mu g t
\]

**At this time:**
\[
 x_1 = \frac{12v_0^2}{49 \mu g}
\]

\[
 \dot{x}(t_1) = v_1 = \frac{5}{7} v_0
\]
Rolling and Sliding Billiard Ball

a ball is struck at \( h \) above the center with a horizontal force. When does it start rolling?

the same equations of motion:

\[
\begin{align*}
\ddot{x} &= -\mu g \\
\dot{a}\phi &= \frac{1}{2}\mu g
\end{align*}
\]

initial conditions:

Impulse \( \equiv \int F_x(t) \, dt = \Delta p_x = Mv_0 \)

\[
\int hF_x(t) \, dt = h \cdot \text{impulse} = \Delta L = I\omega_0
\]

\[
hv_0 = \frac{2}{5}a^2\omega_0
\]

Pure rolling without sliding occurs only for:

- \( h = \frac{2}{3}a \), rolls immediately, no sliding (friction)
- \( h < \frac{2}{3}a \), slides, then rolls
- \( h > \frac{2}{3}a \), rolls too fast?!