Rigid bodies - general theory

Consider a system on \( N \) particles with all their relative separations fixed: it has 3 translational and 3 rotational degrees of freedom.

**Motion with One Arbitrary Fixed Point:**

\[
\frac{d\mathbf{r}}{dt} = \mathbf{\omega} \times \mathbf{r}
\]

\[
\frac{d}{dt} \left( \frac{d\mathbf{r}}{dt} \right)_{\text{inertial}} = \mathbf{\omega} \times \mathbf{\omega} \times \mathbf{r}
\]

Kinetic Energy:

\[
T = \frac{1}{2} \sum_p m_p \mathbf{v}_p^2 = \frac{1}{2} \sum_p \mathbf{m}_p (\mathbf{\omega} \times \mathbf{r}_p) \cdot (\mathbf{\omega} \times \mathbf{r}_p) = \frac{1}{2} \sum_p \mathbf{m}_p (\mathbf{\omega} \times \mathbf{\omega})^2 - (\mathbf{\omega} \cdot \mathbf{r}_p)^2
\]

continuum limit:

\[
T = \frac{1}{2} \int d^3r \rho(r)(\mathbf{\omega} \times \mathbf{\omega})^2
\]

It is convenient to define the inertia tensor:

\[
I_{ij} = \sum_p m_p (\delta_{ij} r_p^2 - x_i x_j)
\]

The kinetic energy has a simple form:

\[
T = \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 I_{ij} \omega_i \omega_j
\]

Both \( I \) and \( \mathbf{\omega} \) depend on the choice of the coordinate system, but \( T \) does not!

Angular momentum:

\[
\mathbf{L} = \sum_p m_p \mathbf{r}_p \times \mathbf{v}_p = \sum_p \mathbf{m}_p (\mathbf{\omega} \times \mathbf{r}_p) - \sum_p \mathbf{m}_p \mathbf{r}_p^2 \mathbf{\omega} - (\mathbf{\omega} \cdot \mathbf{r}_p) \mathbf{r}_p
\]

Continuum limit:

\[
\mathbf{L} = \int d^3r \rho(r)(\mathbf{\omega} \times \mathbf{r})
\]

Using the inertia tensor it can be written as:

\[
I_{ij} = \sum_{k=1}^3 I_{ik} \omega_k
\]

\( I, \mathbf{\omega} \) and \( \mathbf{L} \) depend on the choice of the coordinate system!

And it is related to the kinetic energy:

\[
T = \frac{1}{2} \sum_{i=1}^3 L_i \omega_i = \frac{1}{2} \mathbf{L} \cdot \mathbf{\omega}
\]

General motion with No Fixed Point:

\[
\mathbf{r} = \mathbf{R} + \mathbf{r}'
\]

\[
\frac{d}{dt} \left( \frac{d\mathbf{r}}{dt} \right)_{\text{inertial}} = \frac{d}{dt} \left( \frac{d\mathbf{r}}{dt} \right)_{\text{body}} + \mathbf{\omega} \times \mathbf{r}'
\]

\[
0 \text{ for rigid bodies}
\]
Kinetic Energy and Angular Momentum:

- **body-fixed frame** with the origin at the center of mass
- a parallel copy of the inertial frame with the same origin as the body fixed frame - the center-of-mass frame

**Review:**

\[ M\ddot{R} = \sum \mathbf{F}_i^{(e)} \]
\[ T = \frac{1}{2} M \dot{R}^2 + T' \]
\[ \mathbf{L} = \mathbf{R} \times (M\ddot{R}) + L' \]

Formulas for \( T' \) and \( L' \) are identical to the case of a motion with one fixed point!

Inertia tensor

We have to learn how to evaluate the inertia tensor in a body-fixed frame. First, let’s define the inertia tensor in a body-fixed frame with the origin at the center of mass:

\[ I_{ij} = \sum \rho_i (\delta_{ij} x_i^2 - x_i x_j) \]
\[ I_{ij} = \int d^3 \rho(x) (\delta_{ij} x^2 - x_i x_j) \]
\[ I_{ij} = I_{ji} = (I_{ij})^* \]

Then the inertia tensor in a body-fixed frame with the origin displaced by \( \mathbf{a} \) is given by:

\[ I_{ij} = \tilde{I}_{ij} + M(a^2 \delta_{ij} - a_i a_j) \]

**Example (a uniform disk of radius a):**

- moment of inertia about a perpendicular axis through the edge of the disk (would be tedious to calculate directly)
- moment of inertia about a perpendicular axis through the center of the disk (easy to calculate in cylindrical polar coordinates)

**Principal axes** (a coordinate system in which the inertia tensor is diagonal):

\[ I_{ij} = I_i \delta_{ij} \]

all the formulas simplify in such a coordinate system

Diagonalizing a real symmetric matrix:

\[ \sum_{j=1}^{3} I_{ij} \delta_{ij} = \lambda \delta_{ij} \]
\[ \sum_{j=1}^{3} (I_{ij} - \lambda \delta_{ij}) \delta_{ij} = 0 \]

non-trivial solution only if:

\[ \det \left( I_{ij} - \lambda \delta_{ij} \right) = 0 \]

we get 3 eigenvalues, 3 eigenvectors, and we can form the modal matrix:

\[ \mathbf{\Phi} = \begin{bmatrix} \mathbf{e}^{(1)} & \mathbf{e}^{(2)} & \mathbf{e}^{(3)} \end{bmatrix} \]

\[ \mathbf{\Phi}^T \mathbf{\Phi} = \mathbf{\Phi}^{-1} \mathbf{\Phi} = \mathbf{I} \]

new orthonormal basis - principal axes
Modal matrix diagonalizes the inertia tensor:

\[
\mathbf{L}^{-1} \mathbf{L}^T \mathbf{L}^{-1} = \mathbf{L}^T \mathbf{L} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}
\]

we can define a new set of angular velocities:

\[
\mathbf{\omega} = \mathbf{L} \mathbf{\xi} \quad \mathbf{\xi} = \mathbf{L}^{-1} \mathbf{\omega} = \mathbf{L}^T \mathbf{\omega}
\]

projections of \( \mathbf{\omega} \) along the principal axes

Kinetic Energy becomes:

\[
T = \frac{1}{2} \mathbf{\omega}^T \mathbf{I} \mathbf{\omega} = \frac{1}{2} (\mathbf{L}^T \mathbf{\xi})^T \mathbf{I} (\mathbf{L}^T \mathbf{\xi}) = \frac{1}{2} \mathbf{\xi}^T \mathbf{\xi} = \frac{1}{2} \sum_{i=1}^{3} \xi_i \omega_i I_i
\]

Angular Momentum in the new coordinate system:

\[
\mathbf{L}_{new} = \mathbf{L} \mathbf{\xi} \quad \mathbf{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}
\]

Principal moments of inertia:

\[
I_i = \int d^3r \rho(r) \big( r^2 - \mathbf{r} \cdot \mathbf{\omega}(0))^2 \big)
\]

Euler’s Equations

Describing rotational motion: based on FW-27

\[
\frac{d}{dt} \begin{bmatrix} \mathbf{L}^T \mathbf{\omega} \end{bmatrix}_{\text{inertial}} = \Gamma^{\text{inertial}}
\]

valid in any inertial frame but also in the cm frame:

\[
\frac{d}{dt} \begin{bmatrix} \mathbf{L}^T \mathbf{\omega} \end{bmatrix}_{\text{body}} + \mathbf{\omega} \times \mathbf{L} = \Gamma^{\text{body}}
\]

in terms of body-fixed observables:

\[
I_s = I_2 \omega_3 - I_3 \omega_2 = \omega_2 \omega_3 (I_1 - I_2)
\]

Euler’s equations:

\[
\begin{align*}
\frac{dL_1}{dt} & = I_2 \omega_3 (I_3 - I_2) + \Gamma_1^{\text{inertial}} \\
\frac{dL_2}{dt} & = I_3 \omega_1 (I_1 - I_3) + \Gamma_2^{\text{inertial}} \\
\frac{dL_3}{dt} & = I_1 \omega_2 (I_2 - I_1) + \Gamma_3^{\text{inertial}}
\end{align*}
\]

Applications - Compound Pendulum

Based on FW-28

a rigid body constrained to rotate about a fixed axis:

we can choose the coordinate system so that:

\[
\mathbf{R} = \mathbf{\hat{e}}_{\text{1}}
\]

ignoring friction, the only torque comes from gravity:

\[
\Gamma_{\text{inertial}} = -Mg \mathbf{\hat{e}}_{\text{3}}
\]

in both inertial and body frame

\[
\Gamma_{\text{body}} = \sum_{j} m_j \mathbf{r}_j \times \mathbf{g} = \mathbf{MR} \times \mathbf{g}
\]

inertia:

\[
I = \int d^3r \rho(r) (r^2 - x_3 x_3)
\]

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\]
equation of motion of an arbitrary compound pendulum:

\[ \ddot{\phi} = -\Omega^2 \phi \]

for small oscillations, the motion is simple harmonic:

\[ \ddot{\phi} = -\Omega^2 \phi \]

Let’s rewrite it in terms of the moment of inertia about the axis going through the center of mass:

\[ I \equiv I_{33} = \tilde{I}_{33} + Ml^2 \]
\[ I_{33} = M \bar{k}^2 \]

radius of gyration: equivalent radial distance of a point mass M leading to the same moment of inertia

\[ k = (\bar{k}^2 + \bar{l}^2)^{1/2} \]

radius of gyration about the axis through Q

Let the oscillation frequency about the axis through Q:

\[ \Omega_2 = \frac{g}{\bar{k}^2 + \bar{l}^2} = \frac{g}{l_1} \]
\[ l_1 = l + \frac{k^2}{I} \]

\[ l_1 > l \]

Response of a baseball bat to a transverse force: (we neglect gravity)

reaction force at the point of support

motion of the center of mass (N-2nd):

\[ M\ddot{\mathbf{R}} = \mathbf{f} - \mathbf{f}_s \]
\[ \mathbf{R} \cdot \ddot{\mathbf{e}} = f \dot{\phi} \]

the only direction R can move due to constraints

torque equation:

\[ \frac{dL_3}{dt} = I_3 \ddot{\phi} = \Gamma^G_3 = ft \]
\[ t \phi = M(\bar{k}^2 + \bar{l}^2) \dot{\phi} = ft \]

if the force is applied at the center of percussion
the reaction force at the point of support vanishes!

if applied beyond that point, the reaction force has the same direction as applied force

Rolling and Sliding Billiard Ball

a ball is struck at the center (h=0) with a horizontal force. When does it start rolling?

initial conditions:

\[ x = 0, \dot{x} = v_0, \phi = 0, \dot{\phi} = 0 \]

force equation:

\[ f_r = -\mu g M \dot{\phi} \]

solution:

\[ \ddot{x} = v_0 - \mu g t \]

torque equation:

\[ \frac{dL_3}{dt} = I_3 \ddot{\phi} = \Gamma^G_3 = F_r a \]
\[ a \frac{\ddot{\phi}}{\dot{\phi}} = \frac{5}{2} \mu g t \]

Pure rolling without sliding occurs only for:

\[ I_1 = \frac{2v_0}{7\mu g} \]

at this time:

\[ x_1 = 12v_0^2/49\mu g, \dot{x}(t_1) = v_1 = \frac{2}{7}v_0 \]
Rolling and Sliding Billiard Ball

When does it start rolling?

The ball is struck at \( h \) above the center with a horizontal force.

The same equations of motion:

\[
\dot{x} = -\mu g \\
\dot{\phi} = \frac{3}{2} \mu g
\]

Initial conditions:

\[
\text{Impulse } = \int F_x(t) \, dt = \Delta P_x = Mv_0 \\
\text{Impulse } = \int hF_x(t) \, dt = h \cdot \text{impulse } = \Delta L = I\omega_0
\]

\[
h\omega_0 = \frac{3}{2} M a^2
\]

Pure rolling without sliding occurs only for:

- \( h = \frac{3}{2} a \) rolls immediately, no sliding (friction)
- \( h < \frac{3}{2} a \) slides, then rolls
- \( h > \frac{3}{2} a \) rolls too fast??

Initial angular velocity

\[
I_s = \frac{3}{2} M a^2
\]