Central Forces

Motion of a particle with mass \( m \) subject to a central force:

conservative
\[ \nabla \times F = 0 \]
\[ \nabla \times (f\vec{v}) = (\nabla f) \times \vec{v} + f(\nabla \times \vec{v}) \]

no torque, \( \vec{r} \times F = 0 \) \[ \implies \vec{l} = \vec{r} \times \vec{p} = \text{const.} \]
(motion is planar, perpendicular to \( \vec{l} \))

It is convenient to work with polar coordinates:

\[ x = r \cos \phi \quad \text{and} \quad y = r \sin \phi \]
Conservation of energy:

\[ E = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + V(r) \]

\[ E = T + V = \frac{1}{2}m(r^2 + r^2\dot{\phi}^2) + V(r) \]

(first integral involves only 1st time derivatives)

(homework, problem 1.6)

Conservation of angular momentum:

\[ l = r \times p = \text{const.} \]

\[ l_z = xp_y - yp_x = m(xy - y\dot{x}) \]

\[ l = mr^2\dot{\phi} \]

motion in a small interval of time \( dt \):

\[ d\phi = \dot{\phi} \, dt \]

\[ dA = \frac{1}{2}r(r \, d\phi) = \frac{1}{2}r^2\dot{\phi} \, dt = (l/2m) \, dt \]

\[ \dot{A} = \frac{l}{2m} = \text{const} \]
Kepler’s second law:

An imaginary line drawn from each planet to the Sun sweeps out equal areas in equal times.

(a simple consequence of the conservation of the angular momentum for central forces)
Effective potential:

\[ E = \frac{1}{2} m \dot{r}^2 + V_{\text{eff}}(r) \]

Analyzing the motion:

One-dimensional potential:

\[ V_{\text{eff}}(r) = V(r) + \frac{l^2}{2mr^2} \]

The angular momentum piece acts as repulsive potential, centrifugal barrier.

Turning points:

\[ E - V_{\text{eff}} = \frac{1}{2} m \dot{r}^2 \geq 0 \]

\[ E \geq V_{\text{eff}} \]

\[ E = V_{\text{eff}}(r_0) \]

Minimum energy, circular orbit.
Solution to the trajectory:

\[ E = \frac{1}{2}mr^2 + V_{\text{eff}}(r) \]

Invert to find \( r(t) \)

\[ t = \pm \left( \frac{1}{2}m \right)^{1/2} \int^r dr \left[ E - V_{\text{eff}}(r) \right]^{-1/2} + t_0 \]

Plug in \( r(t) \) to find \( \phi(t) \)

\[ \phi = l m^{-1} \int^t dt \left[ r(t) \right]^{-2} + \phi_0 \]

In practice this is often too complicated, but we can always find the geometric orbit \( r(\phi) \) (at least numerically):

\[ E = \frac{1}{2}mr^2 + V_{\text{eff}}(r) = \frac{l^2}{2mr^4} \left( \frac{dr}{d\phi} \right)^2 + V_{\text{eff}}(r) \]

\[ \dot{r} = (dr/d\phi)\phi = (dr/d\phi)(l/mr^2) \]

\[ l = mr^2 \phi \]

Invert to find \( r(\phi) \)

\[ \phi = \pm l(2m)^{-1/2} \int^r dr \frac{r^{-2}}{r} \left[ E - V_{\text{eff}}(r) \right]^{-1/2} + \phi_0 \]
Kepler’s problem (planetary motion):

Special case when the central force is gravitational force:

From the general solution:

$$V(r) = -\gamma r^{-1}$$

$$\gamma \equiv GM$$

we find:

$$u = r^{-1}$$

$$V_{\text{eff}}(r) = V(r) + \frac{l^2}{2mr^2}$$

$$\phi = \phi_0 \pm \arccos \frac{1 - ul^2/m^2\gamma}{(1 + 2El^2/m^3\gamma^2)^{1/2}}$$

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \begin{cases} \frac{1}{\sqrt{a}} \ln \left( \frac{2\sqrt{a} \sqrt{ax^2 + bx + c} + 2ax + b}{\sqrt{b^2 - 4ac}} \right) \\ -\frac{1}{\sqrt{-a}} \sin^{-1} \left( \frac{2ax + b}{\sqrt{b^2 - 4ac}} \right) \\ \frac{1}{\sqrt{a}} \sinh^{-1} \left( \frac{2ax + b}{\sqrt{4ac - b^2}} \right) \end{cases}$$

$$\arccos x = \frac{\pi}{2} - \arcsin x$$
straightforward manipulations:

\[ u = r^{-1} \]

\[ r^{-1} = C[1 - \epsilon \cos (\phi - \phi_0)] \]

can be set to 0

**Orbit solution:**

For negative energy, orbits are bound and represent an ellipse.
Conic sections:

<table>
<thead>
<tr>
<th>conic section</th>
<th>equation</th>
<th>eccentricity (e)</th>
</tr>
</thead>
<tbody>
<tr>
<td>circle</td>
<td>$x^2 + y^2 = a^2$</td>
<td>0</td>
</tr>
<tr>
<td>ellipse</td>
<td>$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$</td>
<td>$\sqrt{1 - \frac{b^2}{a^2}}$</td>
</tr>
<tr>
<td>parabola</td>
<td>$y^2 = 4ax$</td>
<td>1</td>
</tr>
<tr>
<td>hyperbola</td>
<td>$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$</td>
<td>$\sqrt{1 + \frac{b^2}{a^2}}$</td>
</tr>
</tbody>
</table>
**Ellipse:**

.constant total distance from two foci located at x=+f and x=-f:

\[ d + d' = 2a \]

\[ d = \epsilon a \]

**eccentricity**

\[ f = \sqrt{a^2 - b^2} \]

\[ e = \epsilon = \sqrt{\frac{a^2 - b^2}{a^2}} = \sqrt{1 - \left(\frac{b}{a}\right)^2} = f/a \]

**equivalent definition:**

\[ \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1. \]

in polar coordinates with the origin at the **left** focus:

\[ r(1 - \epsilon \cos \phi) = a(1 - \epsilon^2) \]

follows from the law of cosines:

\[ d^2 = d'^2 - 4d'f \cos \phi + 4f^2 \]

\[ c^2 = a^2 + b^2 - 2ab \cos \gamma \]
Orbit solution:

\[ r^{-1} = C(1 - \epsilon \cos \phi) \]

\[ \epsilon = \left(1 + \frac{2El^2}{m^3\gamma^2}\right)^{1/2} \]

\[ C = \frac{m^2\gamma}{l^2} \]

\[ \gamma \equiv GM \]

Kepler’s first law:
The orbit of a particle with negative energy in a gravitational field is an ellipse with the center at one focus.

\[ e = \epsilon = \sqrt{\frac{a^2 - b^2}{a^2}} = \sqrt{1 - \left(\frac{b}{a}\right)^2} = f/a \]

\[ f = \sqrt{a^2 - b^2} \]

Energy of an orbit does not depend on angular momentum:

\[ E = -\frac{m\gamma}{2a} = -\frac{MmG}{2a} \]
Kepler’s third law:
The square of an orbital period is proportional to the cube of a.

\[ \tau = 2\pi a^{3/2} \gamma^{-1/2} \]
Comment on general central potential:

orbits typically not closed:

\[ \Delta \phi = l(2m)^{-1/2} \int_{r_{\text{min}}}^{r_{\text{max}}} dr \, r^{-2} [E - V_{\text{eff}}(r)]^{-1/2} \]

if rational multiple of \( \pi \) then orbit is closed, otherwise it is open

Harmonic oscillator:

\[ V(r) \propto r^2 \]

Orbit is an ellipse
with the origin at the center!

(homework, problem 1.10)
Two-body motion with a central potential

Everything we have derived for the motion of a particle with mass $m$ subject to a central force:

$$F(r) = \hat{r}f(r)$$

$$F = -\nabla V(r)$$

can be directly (with small modifications) applied to a system of two particles interacting through a central potential:

$$V(|r_1 - r_2|)$$
Newton’s 2nd law:

Using center of mass formulas:

The center of mass moves without acceleration!
Translation from general expressions:

**gravitational force:**

\[ V(r) = -m_1 \frac{\gamma}{r} \]

\[ \gamma \equiv GM \]

\[ \ddot{r} = -\gamma \frac{r}{\mu \ r^3} = -\frac{\gamma}{r^3} \]

\[ \ddot{\gamma} = \frac{\gamma m_1}{\mu} = G(m_1 + m_2) \]

**particle with the reduced mass \( \mu \) subject to a central force**

\[ \mu = \frac{m_1 m_2}{m_1 + m_2} \]

**replacement in all formulas**

**e.g. Kepler’s third law:**

\[ \tau = 2\pi a^{3/2} \gamma^{-1/2} \]

\[ \tau = 2\pi a^{3/2} \gamma^{-1/2} = 2\pi a^{3/2} [G(m_1 + m_2)]^{-1/2} \]

the period also depends on the mass of the planet
Kinetic energy of a two body system:

internal motion about the CM

\[ T = \frac{1}{2} \sum m_i v_i^2 \]

Using center of mass formulas:

\[ r_i = R + r'_i \]

\[ r_1 = R + \frac{m_2 r}{M} \]

\[ r_2 = R - \frac{m_1 r}{M} \]

Angular momentum of a two body system about the center of mass:

the internal angular momentum about the center of mass

\[ L' = \sum m_i r'_i \times v_i \]

Using center of mass formulas:

\[ r_i = R + r'_i \]

\[ r_1 = R + \frac{m_2 r}{M} \]

\[ r_2 = R - \frac{m_1 r}{M} \]

\[ L' = \mu r \times \dot{r} = \mu r \times v \]