Central Forces

Motion of a particle with mass $m$ subject to a central force:

$$ \mathbf{F}(r) = \hat{r} f(r) $$

Conservative:

$$ \nabla \times \mathbf{F} = 0 \quad \nabla \times (f\hat{r}) = (\nabla f) \times \hat{r} + f(\nabla \times \hat{r}) $$

no torque, $\mathbf{r} \times \mathbf{F} = 0$ \quad $l = \mathbf{r} \times \mathbf{p} = \text{const.}$

(motion is planar, perpendicular to $l$)

It is convenient to work with polar coordinates:

$$ x = r \cos \phi \quad \text{and} \quad y = r \sin \phi $$

Conservation of energy:

$$ E = \frac{1}{2}m(\dot{r}^2 + \dot{\phi}^2) + V(r) $$

Conservation of angular momentum:

$$ l = \mathbf{r} \times \mathbf{p} = \text{const.} $$

motion in a small interval of time $dt$:

$$ d\phi = \dot{\phi} \, dt $$

Effective potential:

$$ E = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r) $$

Analyzing the motion:

$$ E - V_{\text{eff}} = \frac{1}{2}m\dot{r}^2 \geq 0 $$

Kepler’s second law:

An imaginary line drawn from each planet to the Sun sweeps out equal areas in equal times.

(a simple consequence of the conservation of the angular momentum for central forces)
Solution to the trajectory:

\[ E = \frac{1}{2} m r^2 + V_{\text{eff}}(r) \]

\[ t = \pm \left( \frac{1}{2m} \right)^{1/2} \int dr \left[ E - V_{\text{eff}}(r) \right]^{-1/2} + t_0 \]

invert to find \( r(t) \)

\[ l = mr^2 \dot{\phi} \]

\[ \phi = \pm \left( \frac{1}{2m} \right)^{1/2} \int dr \left[ E - V_{\text{eff}}(r) \right]^{-1/2} + \phi_0 \]

invert to find \( r(\phi) \)

In practice this is often too complicated, but we can always find the geometric orbit \( r(\phi) \) (at least numerically):

\[ l = mr^2 \dot{\phi} \]

\[ \dot{r} = (dr/d\phi) \dot{\phi} = (dr/d\phi)(l/mr^2) \]

\[ E = \frac{1}{2} m r^2 + V_{\text{eff}}(r) = \frac{l^2}{2mr^4} \left( \frac{dr}{d\phi} \right)^2 + V_{\text{eff}}(r) \]

\[ \phi = \pm \left( \frac{1}{2m} \right)^{1/2} \int dr \ r^{-2} \left[ E - V_{\text{eff}}(r) \right]^{-1/2} + \phi_0 \]

plug in \( r(t) \) to find \( \phi(t) \)

Kepler’s problem (planetary motion):

Special case when the central force is gravitational force:

\[ V(r) = -\frac{\gamma m}{r} \]

\[ \gamma = GM \]

from the general solution:

\[ \phi = \pm \left( \frac{1}{2m} \right)^{1/2} \int dr \ r^{-2} \left[ E - V_{\text{eff}}(r) \right]^{-1/2} + \phi_0 \]

we find:

\[ u = r^{-1} \]

\[ V_{\text{eff}}(r) = V(r) + \frac{\gamma}{2m^2} \]

we find:

\[ \phi = \phi_0 + \int du \left( \frac{2mE}{u^2} + \frac{2m^2\gamma u}{u^2} - u \right)^{-1/2} \]

Conic sections:

For negative energy, orbits are bound and represent an ellipse.
Ellipse:

\[ r(1 - \epsilon \cos \phi) = a(1 - \epsilon^2) \]

constant total distance from two foci located at \(x=+f\) and \(x=-f\):

\[ d + d' = 2a \]

\[ \epsilon = \frac{f}{a} \]

eccentricity

\[ f = \sqrt{a^2 - b^2}. \]

in polar coordinates with the origin at the left focus:

\[ r(1 - \epsilon \cos \phi) = a(1 - \epsilon^2) \]

follows from the law of cosines:

\[ d^2 = d'^2 - 4df \cos \phi + 4f^2 \]

Kepler’s third law:
The square of an orbital period is proportional to the cube of a.

\[ A = \pi \alpha^2 = \int_0^t dtA = \frac{lt}{2m} \]

Kepler’s 2nd law:

\[ A = \frac{l}{2m} = \text{const} \]

Kepler’s 1st law:
The orbit of a particle with negative energy in a gravitational field is an ellipse with the center at one focus.

\[ \epsilon = \left(1 - \frac{2mE}{\gamma^2} \right)^{1/2} \quad C = \frac{m^2 \gamma}{\beta^2} \quad \gamma = GM \]

Kepler’s 3rd law

\[ \frac{r^3 - 2}{t^2} \]

Comment on general central potential:

orbits typically not closed:

\[ \Delta \phi = l(2m)^{-1/2} \int_{\phi_{\max}}^\phi dt \frac{r^2}{2} [E - V_{\text{at}}(r)]^{-1/2} \]

if rational multiple of \( \pi \) then orbit is closed, otherwise it is open

Harmonic oscillator:

\[ V(r) \propto r^2 \]

Orbit is an ellipse with the origin at the center!

(homework, problem 1.10)
Two-body motion with a central potential

Everything we have derived for the motion of a particle with mass $m$ subject to a central force:

$$ F(r) = -\frac{dV(r)}{dr}, \quad F = -\nabla V(r). $$

can be directly (with small modifications) applied to a system of two particles interacting through a central potential:

$$ V(r_1 - r_2) $$

Newton's 2nd law:

$$ m_1 \ddot{r}_1 = -\frac{dV(r_{12})}{dr_1} = -V(r_{12}) \frac{r_2 - r_1}{r_{12}} $$

$$ m_2 \ddot{r}_2 = -\frac{dV(r_{12})}{dr_2} = -V(r_{12}) \frac{r_1 - r_2}{r_{12}} $$

Using center of mass formulas:

$$ \ddot{r}_i = \ddot{R} + \frac{m_i}{M} \dot{r}_i $$

$$ \dot{R} = \frac{1}{M} \sum m_i \dot{r}_i $$

$$ M \ddot{R} = 0 $$

The center of mass moves without acceleration!

Translation from general expressions:

gravitational force:

$$ V(r) = -\frac{m_1 \gamma}{r} $$

$$ \gamma = GM $$

$$ r = -\frac{m_1}{\mu} r $$

$$ \frac{\gamma}{\mu} = G(m_1 + m_2) $$

e.g. Kepler's third law:

$$ \tau = 2\pi a^{3/2} \gamma^{-1/2} $$

the period also depends on the mass of the planet.

Kinetic energy of a two body system:

internal motion about the CM

$$ T = \frac{1}{2} \sum \frac{m_i v_i^2}{M} $$

Particle with the reduced mass $\mu$ subject to a central force:

$$ \ddot{r} = -\frac{dV(r)}{dr} = -V(r) \hat{r} $$

Angular momentum of a two body system about the center of mass:

the internal angular momentum about the center of mass

$$ L = \sum m_i \dot{r}_i \times \dot{v}_i $$

$$ L' = \mu \dot{r} \times \ddot{r} = \mu \dot{r} \times v $$

$$ r_i = R + r'_i $$

$$ r_1 = R + \frac{m_1 r_1}{M} $$

$$ r_2 = R - \frac{m_2 r_1}{M} $$

Using center of mass formulas: