Inverse problem:

Find the force law for a central force that allows a particle to move in a logarithmic spiral orbit given by $r = k e^{\alpha \phi}$, where $k$ and $\alpha$ are constants.
Scattering

Orbit solution for gravitational potential:

\[ r^{-1} = C(1 - \epsilon \cos \phi) \]

\[ \epsilon = \left(1 + \frac{2E l^2}{\mu^3 \bar{\gamma}^2}\right)^{1/2} \]

\[ C = \frac{\mu^2 \bar{\gamma}}{l^2} \]

\[ \bar{\gamma} = \frac{\gamma m_1}{\mu} = G(m_1 + m_2) \]

For positive energy, orbits are unbounded and represent a hyperbola.
Hyperbola:

constant difference in distance from two foci located at x=+f and x=-f:

\[ d - d' = \pm 2a \]

left branch

right branch
eccentricity: \( \epsilon = \frac{f}{a} > 1 \)

asymptotes of the two branches:
\[ \cos \alpha = \epsilon^{-1} \]

equivalent definition:
\[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \]

impact parameter
geright branch in polar coordinates with the origin at the right focus:
\[ r \left(1 - \epsilon \cos \phi\right) = a(\epsilon^2 - 1) \]

follows from the law of cosines:
\[ d'^2 = d^2 + 4f^2 - 4df \cos(\pi - \phi) \]

\[ c^2 = a^2 + b^2 - 2ab \cos \gamma \]

\[ \epsilon = \sqrt{\frac{a^2 + b^2}{a^2}} = \sqrt{1 + \left(\frac{b}{a}\right)^2} = f/a \]
Orbit solution with positive $E$:

$$r^{-1} = C(1 - \epsilon \cos \phi)$$

$$\epsilon = \left(1 + \frac{2E}{\mu^2}\right)^{1/2}$$

$$C = \frac{\mu^2 \bar{\gamma}}{\ell^2}$$

$$\bar{\gamma} = \gamma m_1 = G(m_1 + m_2)$$

is the right branch of a hyperbola with the origin at the right focus!

It is convenient to work with impact parameter $b$ and relative speed at infinity:

$$T' = \frac{1}{2} \mu \dot{r}^2 = \frac{1}{2} \mu \dot{v}^2$$

$$L' = \mu r \times \dot{r} = \mu r \times \dot{v}$$

$$E = \frac{1}{2} \mu \dot{v}^2_{\infty}$$

$$l = \mu v_{\infty} b$$

$$\epsilon = \left[1 + \left(\frac{v_{\infty}^2 b}{\bar{\gamma}}\right)^2\right]^{1/2}$$

distance of closest approach:

$$r_{\text{min}} = f - a = (\epsilon - 1)a = \left(\frac{\epsilon - 1}{\epsilon + 1}\right)^{1/2} b$$

Hyperbola in polar coordinates with the origin at the right focus:

$$r \left(1 - \epsilon \cos \phi\right) = a \left(\epsilon^2 - 1\right)$$

$$\epsilon = \sqrt{\frac{a^2 + b^2}{a^2}} = \sqrt{1 + \left(\frac{b}{a}\right)^2} = f/a$$

$$\cos \alpha = \epsilon^{-1}$$

$$a^2 + b^2 = f^2$$

$$V(r) = -\mu \bar{\gamma}/r.$$
Total deflection angle:

\[ \theta = \pi - 2\alpha \]

\[ \cos \alpha = \epsilon^{-1} \]

\[ \cot \frac{1}{2}\theta = \cot \left( \frac{1}{2}\pi - \alpha \right) = \tan \alpha = (\epsilon^2 - 1)^{1/2} \]

\[ \epsilon = \left[ 1 + \left( \frac{v_{\infty}^2 b}{\bar{\gamma}} \right)^2 \right]^{1/2} \]

\[ \bar{\gamma} = \frac{\gamma m_1}{\mu} = G(m_1 + m_2) \]

\[ \cot \frac{1}{2}\theta = \frac{v_{\infty}^2 b}{\bar{\gamma}} \]

small deflections for large impact parameter and speed and vice versa
Scattering orbits for general central potential (not hyperbolas):

**attractive**

- the distance of closest approach
- corresponding angle

**deflection angle**

- $\phi_m < \pi/2 \quad \Rightarrow \quad \theta > 0$
- $\phi_m > \pi/2 \quad \Rightarrow \quad \theta > 0$

**Formal solution for a given potential:**

\[
\phi_m = l \int_{r_{\min}}^{r_{\infty}} dr \frac{r^2}{r'^2} 
\left[ 2\mu E - 2\mu V(r) - \frac{l^2}{r^2} \right]^{-1/2} + \pi
\]

\[
E = \frac{1}{2} \mu v_{\infty}^2
\]

\[
l = \mu v_{\infty} b
\]
Cross section:

- **incident flux**: (number of particles crossing unit transverse area per unit time)

- **differential cross section**: (constant of proportionality)

- **event p = elastic scattering through a deflection angle between \( \theta \) and \( \theta + d\theta \):**

\[
F 2\pi b \, db = F \, d\sigma_{el}(\theta) = F \left( \frac{d\sigma}{d\Omega} \right)_{el} \, d\Omega
\]

- **differential cross section**: (per unit solid angle)

\[
\left( \frac{d\sigma}{d\Omega} \right)_{el} = \frac{b}{\sin \theta} \left| \frac{db}{d\theta} \right|
\]

- **total cross section**:

\[
\sigma_T = \int d\Omega \left( \frac{d\sigma}{d\Omega} \right)_{el}
\]
Rutherford scattering:

(the scattering of a particle with charge $ze$ and mass $m$ from a nucleus with charge $Ze$ and mass $M$)

Newton’s second law:  
\[ \ddot{r} = \frac{zZe^2r}{r^3} \]

Coulomb’s force

the same as for Kepler problem, hyperbolic gravitational orbits with the replacement:

**Orbit solution:**

\[ r^{-1} = C(1 - \epsilon \cos \phi) \]

\[ \epsilon = \left(1 + \frac{2E\mu}{\mu^2 + \gamma} \right)^{1/2} \]

\[ C = \frac{\mu^2 - \gamma}{L^2} \]

for $C<0$ (repulsive potential):

\[ r^{-1} = |C|\left(\epsilon \cos \phi - 1\right) \]

is the right branch of hyperbola in polar coordinates with the origin at the left focus.
Hyperbola:

**constant difference in distance from two foci** located at \( x=+f \) and \( x=-f \):

\[
d - d' = \pm 2a
\]

**left branch**

**right branch**

**eccentricity:**

\[
\epsilon = \frac{f}{a} > 1
\]

**asymptotes of the two branches:**

\[
\cos \alpha = \epsilon^{-1}
\]

**right branch in polar coordinates with the origin at the right focus:**

\[
r \left(1 - \epsilon \cos \phi \right) = a \left(\epsilon^2 - 1\right)
\]

follows from the law of cosines:

\[
d''^2 = d^2 + 4f^2 - 4df \cos (\pi - \phi)
\]

**right branch in polar coordinates with the origin at the left focus:**

\[
r \left(\epsilon \cos \phi - 1\right) = a \left(\epsilon^2 - 1\right)
\]

follows from the law of cosines:

\[
d^2 = d''^2 + 4f^2 - 4d' f \cos \phi
\]
for positive $C$ we found:

$$\cot \frac{1}{2}\theta = \frac{v^2_{\infty} b}{\gamma}$$

$b$ is identical for both signs of the potential

relation between $b$ and the scattering angle:

$$b = \frac{|zZ| e^2}{\mu v^2_{\infty}} \cot \frac{1}{2}\theta$$

Rutherford formula:

$$\left(\frac{d\sigma}{d\Omega}\right)_{el} = \frac{b}{\sin \theta} \left| \frac{db}{d\theta} \right|$$

$$E = \frac{1}{2} \mu v^2_{\infty}$$

even the total cross section is divergent for small scattering angles!

What does it mean?
Scattering by a hard sphere:

**differential cross section:**

\[
\left( \frac{d\sigma}{d\Omega} \right)_{el} = \frac{b}{\sin \theta} \left| \frac{db}{d\theta} \right|
\]

\[\left( \frac{d\sigma}{d\Omega} \right)_{el} = \frac{1}{4} a^2\]

isotropic

**total cross section:**

\[\sigma_T = \int d\Omega \left( \frac{d\sigma}{d\Omega} \right)_{el} = \pi a^2\]

Infinite for Rutherford-like scattering

**alternatively:**

\[F 2\pi b \, db = F \, d\sigma_{el}(\theta) = F \left( \frac{d\sigma}{d\Omega} \right)_{el} \, d\Omega\]

\[\sigma_T = 2\pi \int_0^{b_{max}} b \, db = \pi b_{max}^2\]

\[\theta = 0 \text{ for } b > a\]