Constrained motion and generalized coordinates

Based on FW-13

Often, the motion of particles is restricted by constraints, and we want to:
- work only with independent degrees of freedom (coordinates)
- eliminate forces of constraint

Motion of $N$ particles, $n = 3N$ degrees of freedom, subject to $k$ equations relating coordinates:

$$ f_j(x_1, \ldots, x_n, t) = c_j \quad j = 1, 2, \ldots, k $$

constraints correspond to forces

holonomic constraints

can be time dependent

the system has $n - k = 3N - k$ degrees of freedom!

Generalized coordinates:

$$ x_1 = x_1(q_1, q_2, \ldots, q_{n-k}, t) $$

$$ x_n = x_n(q_1, q_2, \ldots, q_{n-k}, t) $$

cartesian coordinates subject to $k$ constraints

any set of $n - k = 3N - k$ independent coordinates that completely specify the system

change of a cartesian coordinate induced from changes in generalized coordinates in $dt$:

$$ dx_i = \sum_{a=1}^{n-k} \frac{\partial x_i}{\partial q_a} dq_a + \frac{\partial x_i}{\partial t} dt \quad i = 1, \ldots, n $$

partial derivatives! (all other variables are kept constant)

or, in a compact way:

$$ dx_i = \sum_{a=1}^{n-k} \frac{\partial x_i}{\partial q_a} \delta q_a \quad i = 1, \ldots, n $$

Virtual displacements:

$$ \delta x_i = \sum_{a=1}^{n-k} \frac{\partial x_i}{\partial q_a} \delta q_a \quad i = 1, \ldots, n $$

are defined as infinitesimal, instantaneous displacements of the coordinates consistent with the constraints.

D’Alembert’s Principle

Based on FW-14

The forces of constraint do no work under a virtual displacement:

forces of constraint are perpendicular to the direction of motion and thus they do no work

We can rewrite Newton’s 2nd law as:

$$ \dot{p}_i = F_{i}^{\text{app}} + R_i \quad i = 1, \ldots, n $$

or

$$ \sum_i (F_{i}^{\text{app}} + R_i - \ddot{p}_i) \delta x_i = 0 $$

D’Alembert’s principle:

$$ \sum_i (F_{i}^{\text{app}} - \ddot{p}_i) \delta x_i = 0 $$

forces of constraint have disappeared!
Lagrange’s equations

We want to rewrite D’Alembert’s principle in terms of the generalized coordinates.

The applied force piece:

\[ \sum_{i} (F_i^{applied} - \dot{p}_i) \delta x_i = 0 \]

The virtual work done by applied forces under virtual displacement:

\[ \delta W = \sum_{i} F_i \delta x_i = \sum_{\sigma=1}^{n-k} \left( \sum_{i} \frac{\partial F_i}{\partial \dot{q}_\sigma} \right) \delta q_\sigma = \sum_{\sigma=1}^{n-k} Q_\sigma \delta q_\sigma \]

\[ Q_\sigma = \sum_{i} F_i \frac{\partial \delta x_i}{\partial q_\sigma} \]

The virtual work done by applied forces under virtual displacement can be calculated directly from this definition, or from computing the virtual work done by applied forces for virtual displacement along a given generalized coordinate.

Juggling with time derivatives:

\[ \frac{dx_i}{dt} = \dot{x}_i = \sum_{\sigma=1}^{n-k} \frac{\partial \dot{x}_i}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial \dot{x}_i}{\partial t} \]

\[ \dot{x}_i = \dot{x}_i(q_1, \ldots, q_{n-k}; \dot{q}_1, \ldots, \dot{q}_{n-k}; t) \quad i = 1, \ldots, n \]

Only functions of generalized coordinates, generalized velocities and time - all physically independent variables!

(If specified at a given time, subsequent motion of the system is determined)

Useful formulas:

- Partial derivatives are kept constant when taking partial derivatives
- The order of partial derivatives can be interchanged

Now we can rewrite D’Alembert’s principle in terms of the generalized coordinates:

\[ \sum_{i} (F_i^{applied} - \dot{p}_i) \delta x_i = 0 \]

\[ T = \frac{1}{2} \sum m_i \ddot{q}_i = T(q_1, \ldots, q_{n-k}; \dot{q}_1, \ldots, \dot{q}_{n-k}; t) \]

Lagrange’s equations:

\[ \sum_{i} (F_i^{applied} - \dot{p}_i) \delta x_i = 0 \]

Generalized forces:

\[ \sum_{i} F_i \delta x_i = \sum_{i} \left( \sum_{\sigma=1}^{n-k} \frac{\partial F_i}{\partial q_\sigma} \right) \delta q_\sigma = \sum_{\sigma=1}^{n-k} Q_\sigma \delta q_\sigma \]

Kinetic energy:

\[ \sum_{i} \dot{p}_i \delta x_i = \sum_{i} \left( \sum_{\sigma=1}^{n-k} \frac{\partial \dot{p}_i}{\partial q_\sigma} \right) \delta q_\sigma = \sum_{\sigma=1}^{n-k} Q_\sigma \delta q_\sigma \]

\[ \frac{d}{dt} \left( \sum_{i} m_i \dot{q}_i \right) = T(q_1, \ldots, q_{n-k}; \dot{q}_1, \ldots, \dot{q}_{n-k}; t) \]

n-k equations for n-k independent generalized coordinates (equivalent to Newton’s 2nd law)
For conservative forces (potential energy depends only on the position):

\[ V(x_1, \ldots, x_n) = V(q_1, \ldots, q_{n-k}, t) \]

generalized forces are given by negative gradients with respect to corresponding generalized coordinate:

\[ Q_x = \sum_f \frac{\partial V}{\partial x_j} \frac{\partial x_j}{\partial q_x} = -\sum_f \frac{\partial V}{\partial x_j} \frac{\partial x_j}{\partial q_x} \]

\[ = -\frac{\partial}{\partial q_x} V(q_1, \ldots, q_{n-k}, t) \]

Lagrange's equations for conservative forces:

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_x} - \frac{\partial L}{\partial q_x} = 0 \quad \sigma = 1, \ldots, n-k \]

\[ L \equiv T - V \]

Using Lagrange's equations

Pendulum: \( \theta \) is the generalized coordinate

\[ T = \frac{1}{2} m (\dot{\theta})^2 \]

\[ V = -mgL \cos \theta + \text{const} \]

\[ L = \frac{1}{2} mL^2 \dot{\theta}^2 + mgL \cos \theta - \text{const} \]

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \quad \sigma = 1, \ldots, n-k \]

\[ \ddot{\theta} + \frac{g}{L} \sin \theta = 0 \]

Bead on a Rotating Wire Hoop:

\( \theta \) is the generalized coordinate

\[ \begin{align*}
    x &= a \cos \omega t + a \cos (\omega t + \theta) \\
    y &= a \sin \omega t + a \sin (\omega t + \theta)
\end{align*} \]

\[ T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) \]

\[ T = \frac{1}{2} m a^2 (\omega^2 + (\omega + \dot{\theta})^2 + 2\omega (\omega + \dot{\theta}) \cos \theta) \]

\[ \ddot{\theta} + \omega^2 \sin \theta = 0 \]

Moving inside a cone:

A particle of mass \( m \) is constrained to move on the inside surface of a smooth cone of half-angle \( \alpha \). The particle is subject to a gravitational force. Determine a set of generalized coordinates and find Lagrange's equations of motion.
Rotating pendulum: The point of support of a simple pendulum of length $b$ moves on a massless rim of radius $a$ rotating with constant angular velocity $\omega$. Find the equation of motion for the angle $\theta$.

Pendulum with horizontally moving support: A simple pendulum of mass $m_2$ with a mass $m_1$ at the point of support which can move on a horizontal line in the plane in which $m_2$ moves. Find the equation of motion for $x$ and the angle $\phi$. 