Calculus of Variations

Motivation: We will be able to obtain the whole set of Lagrange's equations from a single variational principle.

A functional, a function of \( x, y(x) \) and \( y'(x) \)

\[ y'(x) = \frac{dy}{dx} \]

Problem:

Find the function \( y(x) \) that makes

\[ I \equiv \int_{x_1}^{x_2} \phi(y, y', x) \, dx \]

an extremum (for us minimum).

Examples:

What function \( y(x) \) minimizes the distance between 1 and 2?

\[ ds = \left[ (dx)^2 + (dy)^2 \right]^{1/2} = [1 + (y')^2]^{1/2} \, dx \]

the functional for this problem is:

\[ I \equiv \int_{x_1}^{x_2} \phi(y, y', x) \, dx \]

\[ \phi = [1 + (y')^2]^{1/2} \]

What shape of the wire minimizes the time of travel from point 1 to 2? (no friction, uniform gravitational field)

\[ t_{12} = \int_{1}^{2} \frac{ds}{v} \]

\[ \frac{1}{2} m v^2 = m g y \]

the functional for this problem is:

\[ \phi = \left[ \frac{1 + (y')^2}{2gy} \right]^{1/2} \]

the solution is called a brachistochrone
Problem: Find the function $y(x)$ that makes $I = \int_{x_1}^{x_2} \phi(y, y', x) \, dx$ an extremum.

Solution:

Let $y(x)$ be the solution, and construct infinitesimal arbitrary functions, satisfying: $\eta(x_1) = \eta(x_2) = 0$

Let’s calculate the integral for $Y(x)$:

$$I(\epsilon) = \int_{x_1}^{x_2} \phi(Y(x), Y'(x), x) \, dx = \int_{x_1}^{x_2} \phi(y(x) + \epsilon \eta(x), y'(x) + \epsilon \eta'(x), x) \, dx$$

Taylor series expansion about $\epsilon = 0$

$$I(\epsilon) = \int_{x_1}^{x_2} \phi(y, y', x) + \epsilon \int_{x_1}^{x_2} \left( \frac{\partial \phi}{\partial y} \eta(x) + \frac{\partial \phi}{\partial y'} \eta'(x) \right) \, dx + O(\epsilon^2)$$

$$\left| \frac{dI(\epsilon)}{d\epsilon} \right|_{\epsilon=0} = 0$$

$I(\epsilon)$ has an extremum for $\epsilon = 0$!

Solution (continued):

$$\int_{x_1}^{x_2} \frac{\partial \phi}{\partial y} \eta(x) + \frac{\partial \phi}{\partial y'} \eta'(x) \, dx = 0$$

$$\frac{\partial \phi}{\partial y'} \eta(x) = \frac{d}{dx} \left[ \frac{\partial \phi}{\partial y} \eta(x) \right] - \eta(x) \frac{d}{dx} \frac{\partial \phi}{\partial y'}$$

$$\int_{x_1}^{x_2} \frac{d}{dx} \left[ \frac{\partial \phi}{\partial y} \eta(x) \right] \, dx = \left[ \eta(x) \frac{\partial \phi}{\partial y'} \right]_{x_1}^{x_2} = 0$$

$$\eta(x_1) = \eta(x_2) = 0$$

Euler-Lagrange equation for the variational problem!
Example:

What function \( y(x) \) minimizes the distance between 1 and 2?

\[
ds = [(dx)^2 + (dy)^2]^{1/2} = [1 + (y')^2]^{1/2} \, dx
\]

the functional for this problem is:

\[
I = \int_{x_1}^{x_2} \phi(y, y', x) \, dx
\]

and the solution is obtained from Euler-Lagrange equation:

\[
\frac{d}{dx} \frac{y'}{[1 + (y')^2]^{1/2}} - \frac{(y')^2}{[1 + (y')^2]^{3/2}} = \frac{y''}{[1 + (y')^2]^{3/2}} = 0
\]

\[
y'' = 0
\]

\[
y(x) = ax + b
\]

straight line (as expected)

Connection of what we jus did with variations:

\[
Y(x) = y(x) + \epsilon \eta(x)
\]

infinitesimal

arbitrary functions, satisfying:

\[
\eta(x_1) = \eta(x_2) = 0
\]

\[
I(\epsilon) = \int_{x_1}^{x_2} \phi [Y(x), Y'(x), x] \, dx = \int_{x_1}^{x_2} \phi [y(x) + \epsilon \eta(x), y'(x) + \epsilon \eta'(x), x] \, dx
\]

Taylor series expansion about \( \epsilon = 0 \)

\[
I(\epsilon) = \int_{x_1}^{x_2} \phi(y, y', x) + \epsilon \int_{x_1}^{x_2} \left[ \frac{\partial \phi}{\partial y} \eta(x) + \frac{\partial \phi}{\partial y'} \eta'(x) \right] \, dx + O(\epsilon^2)
\]

variation of the functional:

\[
\phi [Y(x), Y'(x), x] - \phi [y(x), y'(x), x] = \delta \phi
\]

Taylor series expansion:

\[
\delta \phi = \frac{\partial \phi}{\partial y} \delta y + \frac{\partial \phi}{\partial y'} \delta y'
\]
Connection of what we just did with variations:

\[ I(\epsilon) = \int_{x_1}^{x_2} \phi(y', y', x) + \epsilon \int_{x_1}^{x_2} \frac{\partial \phi}{\partial y'} y''(x) + \frac{\partial \phi}{\partial y'} y'(x) \, dx + O(e^2) \]

\[ \left. \frac{dI(\epsilon)}{d\epsilon} \right|_{\epsilon=0} = 0 \]

\[ \int_{x_1}^{x_2} \frac{\partial \phi}{\partial y'} y''(x) + \frac{\partial \phi}{\partial y'} y'(x) \, dx = 0 \]

\[ \delta I = 0 \] (partial integration)

Euler-Lagrange equation for the variational problem!

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Hamilton’s principle based on FW-18

\[ I = \int_{x_1}^{x_2} \phi(y', y', x) \, dx \]

\[ \delta y(x_1) = \delta y(x_2) = 0 \]

\[ \delta I = \int_{x_1}^{x_2} \frac{\partial \phi}{\partial y'} \delta y \, dx \]

\[ \delta I = 0 \] (partial integration)

\[ \frac{d}{dx} \frac{\partial \phi}{\partial y'} - \frac{\partial \phi}{\partial y} = 0 \]

Variational statement of mechanics:

(For conservative forces)

\[ \delta \left[ \int_{t_1}^{t_2} L[q(t), \dot{q}(t), t] \, dt \right] = 0 \]

\[ \delta q(t_1) = \delta q(t_2) = 0 \]

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \]

Action

Equivalent to Newton’s laws!

\[ L = T - V \]

The particle takes the path that minimizes the integrated difference of the kinetic and potential energies.
Generalization to a system with \( n \) degrees of freedom:

\[
0 = \delta \int_{t_1}^{t_2} L(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t) \, dt
\]

\[
= \int_{t_1}^{t_2} \sum_{\sigma=1}^{n} \left( \frac{\partial L}{\partial q_\sigma} \delta q_\sigma + \frac{\partial L}{\partial \dot{q}_\sigma} \delta \dot{q}_\sigma \right) \, dt
\]

\[
\delta q_\sigma(t) = \epsilon q_\sigma(t) \quad \sigma = 1, \ldots, n
\]

\[
\delta \dot{q}_\sigma = \frac{d}{dt} \delta q_\sigma
\]

\[
\int_{t_1}^{t_2} \left[ \sum_{\sigma=1}^{n} \delta q_\sigma \left( \frac{\partial L}{\partial q_\sigma} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} \right) \right] \, dt = 0
\]

If all the generalized coordinates are independent:

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0 \quad \sigma = 1, \ldots, n
\]

For \( k \) holonomic constraints:

\[
f_j(q_1, \ldots, q_n; t) = c_j \quad j = 1, \ldots, k
\]

\[
\delta f_j = \sum_{\sigma=1}^{n} \frac{\partial f_j}{\partial q_\sigma} \delta q_\sigma = 0 \quad j = 1, \ldots, k
\]

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q_\sigma} \quad \sigma = 1, \ldots, n
\]

Lagrange multipliers (can be chosen so that coefficients of \( k \) dependent variations of coordinates vanish.

Forces of constraint

Often it is useful to incorporate (some) constraints into Hamilton’s principle:

\[
\int_{t_1}^{t_2} \left[ \sum_{\sigma=1}^{n} \delta q_\sigma \left( \frac{\partial L}{\partial q_\sigma} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} \right) \right] \, dt = 0
\]

\[
f_j(q_1, \ldots, q_n; t) = c_j \quad j = 1, \ldots, k
\]

\[
\delta f_j = \sum_{\sigma=1}^{n} \frac{\partial f_j}{\partial q_\sigma} \delta q_\sigma = 0 \quad j = 1, \ldots, k
\]

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q_\sigma} \quad \sigma = 1, \ldots, n
\]

\[
f_j(q_1, \ldots, q_n; t) = c_j \quad j = 1, \ldots, k
\]
Lagrange multipliers determine reaction forces:

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q_\sigma} \quad \sigma = 1, \ldots, n
\]

\[f_j(q_1, \ldots, q_n, t) = c_j \quad j = 1, \ldots, k\]

\[L = T - V\]

Lagrange’s equations:

\[
\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} = Q_\sigma \quad \sigma = 1, \ldots, n
\]

\[
\delta W = \sum_{\sigma} Q_\sigma \delta q_\sigma
\]

forces of constraint (reaction forces)
(given by Lagrange multipliers)
we can choose to include any one or all constraints, solve \(n+k\) equations for \(n+k\) unknowns, including Lagrange multipliers, and determine reaction forces of interest.

(reaction forces correspond to variations of generalized coordinates that violate the constraints)

Using Lagrange’s equations

Based on FW-16

Pendulum:

\[T = \frac{1}{2} m (l \dot{\theta})^2\]

\[V = -mgl \cos \theta + \text{const}\]

\[L = \frac{1}{2} ml^2 \dot{\theta}^2 + mgl \cos \theta - \text{const}\]

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0 \quad \sigma = 1, \ldots, n-k
\]

\[
\frac{d}{dt} ml^2 \ddot{\theta} - (mgl \sin \theta) = 0
\]

\[
\ddot{\theta} = -\frac{g}{l} \sin \theta
\]

pendulum equation,
small small-amplitude approximation - oscillations with
\[
\omega = (g/l)^{1/2}
\]
Pendulum: (with $r$ and $\theta$ as generalized coordinates)

\[ L = \frac{1}{2}m(r^2 + r^2\dot{\theta}^2) + mgr \cos \theta + \text{const} \]

**constraint:**

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = \sum_{j=1}^{n} \frac{\partial f_j}{\partial q_\sigma} \quad \sigma = 1, \ldots, n \]

\[ \delta r = 0 \]

\[ r = l \]

\[ \dot{r} = \dot{r} = 0 \]

3 eqns. for 3 unknown

\[ m(r - r\dot{\theta}^2) - mg \cos \theta = \lambda = Q_r \]

\[ \frac{d}{dt} mv^2\dot{\theta} + mgr \sin \theta = 0 \]

the force of constraint is the tension:

\[ \delta W = Q_r, \delta r = -\tau, \delta \dot{r} \]

\[ Q_r = \lambda = -\tau \]

tension force given by the the centrifugal force and the $r$-component of the gravitational force

pendulum equation

(reaction forces correspond to variations of generalized coordinates that violate the constraints)