Calculus of Variations

Motivation: We will be able to obtain the whole set of Lagrange’s equations from a single variational principle.

Problem:
Find the function \( y(x) \) that makes

\[
I = \int_{x_1}^{x_2} \phi(y, y', x) \, dx
\]

an extremum (for us minimum).

Examples:
What function \( y(x) \) minimizes the distance between 1 and 2?

\[ds = \left[(dx)^2 + (dy)^2\right]^{1/2} = \left[1 + (y')^2\right]^{1/2} \, dx\]

the functional for this problem is:

\[\phi = \left[1 + (y')^2\right]^{1/2}\]

What shape of the wire minimizes the time of travel from point 1 to 2? (no friction, uniform gravitational field)

\[t_{12} = \int_1^{x_2} \frac{ds}{v} = \int_1^{x_2} \frac{\left[1 + (y')^2\right]^{1/2}}{2gy} \, dx\]

the functional for this problem is:

\[\phi = \frac{1 + (y')^2}{2gy}\]

the solution is called a brachistochrone

Problem: Find the function \( y(x) \) that makes \( I = \int_{x_1}^{x_2} \phi(y, y', x) \, dx \) an extremum.

Solution:
Let \( y(x) \) be the solution, and construct

\[Y(x) = y(x) + \epsilon \eta(x)\]

infinitesimal arbitrary functions, satisfying:
\[\eta(x_1) = \eta(x_2) = 0\]

Let’s calculate the integral for \( Y(x) \):

\[I(\epsilon) = \int_{x_1}^{x_2} \phi(y, y', x) + \epsilon \left[ \frac{\partial \phi}{\partial y} \eta(x) + \frac{\partial \phi}{\partial y'} \eta'(x) \right] \, dx + O(\epsilon^2)\]

Taylor series expansion about \( \epsilon = 0 \)

\[I(\epsilon) \text{ has an extremum for } \epsilon = 0!\]

\[\left. \frac{dI(\epsilon)}{d\epsilon} \right|_{\epsilon=0} = 0\]

Solution (continued):

Problem: Find the function \( y(x) \) that makes \( I = \int_{x_1}^{x_2} \phi(y, y', x) \, dx \) an extremum.

Solution (continued):

\[\int_{x_1}^{x_2} \frac{\partial \phi}{\partial y'} \eta(x) - \frac{\partial \phi}{\partial y} \eta'(x) \, dx = 0\]

arbitrary functions

Euler-Lagrange equation for the variational problem!
Example:

What function \( y(x) \) minimizes the distance between 1 and 2?

\[
ds = \left[ (dx)^2 + (dy)^2 \right]^{1/2} = \left[ 1 + (y')^2 \right]^{1/2} \, dx
\]

the functional for this problem is:

\[
I = \int_{x_1}^{x_2} \phi(y, y', x) \, dx
\]

and the solution is obtained from Euler-Lagrange equation:

\[
\frac{d}{dx} \left( \frac{\partial \phi}{\partial y'} \right) - \frac{\partial \phi}{\partial y} = 0
\]

\[
y(x) = ax + b
\]

straight line (as expected)

Connection of what we jus did with variations:

\[
l = \int_{x_1}^{x_2} \phi(y, y', x) \, dx
\]

\[
\delta l = \int_{x_1}^{x_2} \left( \frac{\partial \phi}{\partial y'} \delta y' + \frac{\partial \phi}{\partial y} \delta y \right) \, dx
\]

\[
\delta l = 0
\]

arbitrary functions

variation of the functional:

\[
\phi[y(x), y'(x), x] - \phi[y(x), y'(x), x] = \delta \phi
\]

Taylor series expansion:

\[
\delta \phi = \frac{\partial \phi}{\partial y'} \delta y' + \frac{\partial \phi}{\partial y} \delta y
\]

Hamilton’s principle

based on FW-18

Variational statement of mechanics:

(for conservative forces)

\[
L = T - V
\]

the particle takes the path that minimizes the integrated difference of the kinetic and potential energies

Equivalent to Newton’s laws!
Generalization to a system with $n$ degrees of freedom:

\[
0 = \delta \int_{t_1}^{t_2} L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n; t) \, dt
\]

\[
= \int_{t_1}^{t_2} \sum_{i=1}^{n} \left( \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i + \frac{\partial L}{\partial q_i} \delta q_i \right) \, dt
\]

\[
= \int_{t_1}^{t_2} \sum_{i=1}^{n} \delta q_i \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} \right) \, dt
\]

\[
\delta q_i(t_1) = \delta q_i(t_2) = 0 \quad \sigma = 1, \ldots, n
\]

\[
\int_{t_1}^{t_2} \sum_{i=1}^{n} \left( \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i - \frac{\partial L}{\partial q_i} \delta q_i \right) \, dt = 0
\]

Forces of constraint

Often it is useful to incorporate (some) constraints into Hamilton’s principle:

\[
\int_{t_1}^{t_2} \sum_{i=1}^{n} \delta q_i \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} \right) \, dt = 0
\]

\[
f(q_1, \ldots, q_n, t) = c_j \quad j = 1, \ldots, k
\]

Lagrange multipliers determine reaction forces:

\[
\delta W = \sum \left( \frac{\partial V}{\partial \delta q} + \left( \begin{array}{c} \frac{\partial V}{\partial \delta q} \\ \cdot \end{array} \right) \right)
\]

Lagrange’s equations:

\[
d \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q_i} \quad \sigma = 1, \ldots, n
\]

Forces of constraint (reaction forces) (given by Lagrange multipliers)

we can choose to include any one or all

constraints, solve $n+k$ equations for $n+k$

unknowns, including Lagrange multipliers,

and determine reaction forces of interest.

(reaction forces correspond to variations of

generalized coordinates that violate the

constraints)

\[
\delta W = \sum Q \delta q
\]

Using Lagrange’s equations

Pendulum:

\[
T = \frac{1}{2}m(l\dot{\theta})^2
\]

\[
V = -mgll \cos \theta + \text{const}
\]

\[
L = T - V
\]

\[
L = \frac{1}{2}ml^2 \dot{\theta}^2 + mgl \cos \theta - \text{const}
\]

\[
\frac{d}{dt} ml^2 \dot{\theta} - (mlg \sin \theta) = 0
\]

\[
\ddot{\theta} = -\frac{g}{l} \sin \theta
\]

pendulum equation, small small-amplitude
approximation - oscillations with

\[
\omega = (g/l)^{1/2}
\]
Pendulum: (with \( r \) and \( \theta \) as generalized coordinates)

\[
L = \frac{1}{2}m(r^2 + r^2\dot{\theta}^2) + mgy\cos \theta + \text{const}
\]

Constraint:

\[
\begin{align*}
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} &= \sum_{j=1}^n \frac{\partial}{\partial \dot{q}_j} \lambda_j \delta r \\
\delta r &= 0
\end{align*}
\]

\[
m(r - r\dot{\theta}^2) - mgy\cos \theta = \lambda = Q_r
\]

\[
\frac{d}{dt} m\dot{\theta}^2 + mgy\sin \theta = 0
\]

3 eqns. for 3 unknowns:

\[
\begin{align*}
r &= l \\
\dot{r} &= \dot{\theta} = 0
\end{align*}
\]

Tension force given by the centrifugal force and the \( r \)-component of the gravitational force:

\[
-\lambda = \tau = m\ddot{\theta}^2 + mgy \cos \theta
\]

\[
\dot{\theta} = -\frac{\dot{\theta}}{l} \sin \theta
\]

The force of constraint is the tension:

\[
\delta W = Q_r \delta r = -\tau \delta r
\]

\[
Q_r = \lambda = -\tau
\]

(reaction forces correspond to variations of generalized coordinates that violate the constraints)