We will consider small-amplitude oscillations of mechanical systems about static equilibrium, e.g. coupled pendulums:

applications: vibrations of molecules, crystals,...
Consider a system described by a set of \( n \) independent generalized coordinates, with time-independent potential and no time-varying constraints:

\[
L = L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)
\]

\[
V = V(q_1, \ldots, q_n)
\]

\[
x_i = x_i(q_1, \ldots, q_n)
\]

equations of motion:

\[
\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} = Q_\sigma = -\frac{\partial V}{\partial q_\sigma} \quad \sigma = 1, \ldots, n
\]

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0 \quad \sigma = 1, \ldots, n
\]

\[
L \equiv T - V
\]

Static equilibrium:

\[
q_\sigma = q_\sigma^0 \quad \dot{q}_\sigma = \ddot{q}_\sigma = 0 \quad \sigma = 1, \ldots, n
\]

all generalized forces have to vanish:

\[
Q_\sigma = -\left( \frac{\partial V}{\partial q_\sigma} \right)_{q^0} = 0 \quad \sigma = 1, \ldots, n
\]

Stability:

if the potential has a minimum, the equilibrium is stable
Consider a **small displacement from equilibrium**:

\[ q_\sigma = q^0_\sigma + \eta_\sigma \quad \sigma = 1, \ldots, n \]

\[ \dot{q}_\sigma = \dot{\eta}_\sigma \quad \sigma = 1, \ldots, n \]

**Kinetic energy:**

\[ T = \frac{1}{2} \sum_\sigma \sum_\lambda m_{\sigma \lambda} \dot{\eta}_\sigma \dot{\eta}_\lambda \]

evaluated at the equilibrium - constant matrix!

\[ m_{\sigma \lambda} = m_{\lambda \sigma} = m^*_{\sigma \lambda} = m^*_{\lambda \sigma} \]

**Potential energy:**

\[ V(q_1, \ldots, q_n) = V(q^0_1, \ldots, q^0_n) + \sum_\sigma \eta_\sigma \left( \frac{\partial V}{\partial q^0_\sigma} \right) + \frac{1}{2} \sum_\sigma \sum_\lambda \eta_\sigma \eta_\lambda \left( \frac{\partial^2 V}{\partial q^0_\sigma \partial q^0_\lambda} \right) \]

\[ Q_\sigma = - \left( \frac{\partial V}{\partial q^0_\sigma} \right) = 0 \quad \sigma = 1, \ldots, n \]

evaluated at the equilibrium - constant matrix!

\[ v_{\sigma \lambda} = \left( \frac{\partial^2 V}{\partial q^0_\sigma \partial q^0_\lambda} \right)q^0 \]

\[ v_{\sigma \lambda} = v_{\lambda \sigma} = v^*_{\sigma \lambda} = v^*_{\lambda \sigma} \]
Lagrangian:

\[ q_{\sigma} = q_{\sigma}^0 + \eta_{\sigma} \quad \sigma = 1, \ldots, n \]

\[ L = T - V = \frac{1}{2} \sum_{\sigma} \sum_{\lambda} (m_{\lambda \sigma} \dot{\eta}_{\lambda} \dot{\eta}_{\sigma} - v_{\lambda \sigma} \eta_{\lambda} \eta_{\sigma}) - V_0 \]

- Quadratic in small displacements and derivatives
- Constant real symmetric matrices

Equations of motion:

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_{\sigma}} - \frac{\partial L}{\partial q_{\sigma}} = 0 \quad \sigma = 1, \ldots, n \]

\[ \sum_{\lambda=1}^{n} (m_{\sigma \lambda} \ddot{\eta}_{\lambda} + v_{\sigma \lambda} \dot{\eta}_{\lambda}) = 0 \quad \sigma = 1, \ldots, n \]

- Linear in small displacements and derivatives
Normal modes - pendulum

Small displacement problem for a system described by 1 generalized coordinate:

Introduce complex coordinate:

We seek a solution of the form:

eigenvalue equation:
General solution:

$$z(t) = z_+^{(1)} e^{i \omega_1 t} + z_-^{(1)} e^{-i \omega_1 t}$$

General solution of the original problem:

$$\eta = \text{Re} \ z = \frac{1}{2} (z + z^*) = \frac{1}{2} \{ [z_+^{(1)} + (z_-^{(1)})^*] e^{i \omega_1 t} + \text{complex conjugate} \}$$

$$z_+^{(1)} + (z_-^{(1)})^* = z^{(1)} = \rho^{(1)} e^{i \phi_1}$$

$$\eta = \rho^{(1)} \cos (\omega_1 t + \phi_1) = \text{Re} \ (z^{(1)} e^{i \omega_1 t}) = \text{Re} \ (\rho^{(1)} e^{i(\omega_1 t + \phi_1)})$$

we wrote the general solution using only positive eigenvalue
In general, we need to solve a set of \( n \) linear homogeneous coupled differential equations with constant coefficients. It is convenient to introduce complex parameters:

\[
\sum_{\lambda=1}^{n} (m_{\sigma \lambda} \ddot{z}_\lambda + v_{\sigma \lambda} z_\lambda) = 0 \quad \sigma = 1, \ldots, n
\]

We will look first for normal modes:

\[
z_\sigma = z_\sigma^0 e^{i\omega t} \quad \sigma = 1, \ldots, n
\]

all the coordinate oscillate with the same frequency
Small-amplitude oscillations of coupled pendulums:

Lagrangian:

\[ L = \frac{1}{2}m(\dot{\eta}_1^2 + \dot{\eta}_2^2) - \frac{mg}{2l} (\eta_1^2 + \eta_2^2) - \frac{1}{2}k(\eta_1^2 + \eta_2^2 - 2\eta_1\eta_2) \]
Lagrange’s equations:

\[ L = \frac{1}{2}m(\dot{\eta}_1^2 + \dot{\eta}_2^2) - \frac{mg}{2l}(\eta_1^2 + \eta_2^2) - \frac{1}{2}k(\eta_1^2 + \eta_2^2 - 2\eta_1\eta_2) \]

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{\eta}_\sigma} - \frac{\partial L}{\partial \eta_\sigma} = 0 \quad \sigma = 1, \ldots, n \]

\[ \sum_{\lambda=1}^{n} (m_{\sigma \lambda} \ddot{\eta}_\lambda + v_{\sigma \lambda} \eta_\lambda) = 0 \quad \sigma = 1, \ldots, n \]

Solving for normal modes:

\[ \eta_\sigma = C \rho_\sigma \cos (\omega t + \phi) \quad \sigma = 1, 2 \]

both pendulums oscillate with the same frequency

in matrix notation

\[ ((v - m\omega^2) \rho) = 0 \]

\[ \rho = \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} \]

\[ v = \begin{bmatrix} \frac{mg}{l} + k & -k \\ -k & \frac{mg}{l} + k \end{bmatrix} \]

\[ m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]
Linear equations - math review:

Consider a set of \( n \) linear inhomogeneous equations (real coefficients):

\[
\begin{align*}
 a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &= y_1 \\
 a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n &= y_2 \\
 &\vdots \\
 a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n &= y_n
\end{align*}
\]

Solution:

If \( \det a \neq 0 \) then
Consider a set of \( n \) linear homogeneous equations:

\[
\sum_{j=1}^{n} a_{ij} x_j = y_i \quad i = 1, \ldots, n
\]

Solution:

- If \( \det A \neq 0 \) then

  
  
  at least one equation is linearly dependent, and can be discarded. Then, assuming the \( n \)-th component of \( x \) is non-zero, we can divide all remaining equations by it...

- If \( \det A = 0 \) then

  
  only trivial solution

and obtain a set of \( n-1 \) inhomogeneous equations

\[
\begin{align*}
\frac{x_1}{x_n} + \frac{a_{12}}{x_n} + \cdots + \frac{a_{1,n-1}}{x_n} &= -a_{1n} \\
&
\vdots \\
\frac{a_{n-1,1}}{x_n} + \cdots + \frac{a_{n-1,n-1}}{x_n} &= -a_{n-1,n}
\end{align*}
\]
Back to coupled pendulums:

The system has a non-trivial solution only if:

\[ ax = 0 \]
\[ \det a = 0 \]

Two solutions for normal-mode frequencies:

\[ \left( m\omega_1^2 - \frac{mg}{l} \right) \left[ m\omega_2^2 - \left( \frac{mg}{l} + 2k \right) \right] = 0 \]

\[ \omega_1 = \left( \frac{g}{l} \right)^{1/2} \]
\[ \omega_2 = \left( \frac{g}{l} + 2 \frac{k}{m} \right)^{1/2} \]

- Free pendulum
- Higher frequency
**Corresponding normal-mode eigenvectors:**

\[
\omega_1 = \left(\frac{g}{l}\right)^{1/2}
\]

\[
k\rho_1^{(1)} - k\rho_2^{(1)} = 0
\]

\[
\rho_1^{(1)} = +\rho_2^{(1)}
\]

\[
\omega_2 = \left(\frac{g}{l} + 2 \frac{k}{m}\right)^{1/2}
\]

\[
-k\rho_1^{(2)} - k\rho_2^{(2)} = 0
\]

\[
\rho_1^{(2)} = -\rho_2^{(2)}
\]

\[
\eta_\sigma = C\rho_\sigma \cos \left(\omega t + \phi\right) \quad \sigma = 1, 2
\]
Our set of linear homogeneous equations has a nontrivial solution only if:

\[ \det |v_{\sigma \lambda} - \omega^2 m_{\sigma \lambda}| = 0 \]

This leads to an n-th order polynomial, which has n roots:

\[ \sum_{\lambda=1}^{n} (v_{\sigma \lambda} - \omega^2 m_{\sigma \lambda}) z_{\lambda}^{(s)} = 0 \]

All the roots are real

\[ \sum_{\sigma} z_{\sigma}^{(s)} = \frac{\sum_{\sigma} \sum_{\lambda} z_{\sigma}^{(s)*} v_{\sigma \lambda} z_{\lambda}^{(s)}}{\sum_{\sigma} \sum_{\lambda} z_{\sigma}^{(s)*} m_{\sigma \lambda} z_{\lambda}^{(s)}} \]

\[ \omega_s^2 = \frac{m_{\sigma \lambda} = m_{\lambda \sigma} = m_{\sigma \lambda}^* = m_{\lambda \sigma}^*}{v_{\sigma \lambda} = v_{\lambda \sigma} = v_{\sigma \lambda}^* = v_{\lambda \sigma}^*} \]

\[ (\omega_s^2)^* = \omega_s^2 \quad s = 1, \ldots, n \]

given non-trivial solution

differs only by interchange of dummy summation indices
Stability:

- \( \omega_s^2 < 0 \) unstable
- \( \omega_s^2 \geq 0 \) stable

Imaginary frequency leads to runaway solutions:

\[ z_\sigma = z_0 e^{i\omega t} \quad \sigma = 1, \ldots, n \]
\[ \eta_\sigma = \text{Re} z_\sigma \quad \sigma = 1, \ldots, n \]

Form of the general solution:

\[ \omega_s^2 = \sum \frac{z_\sigma^{(s)} v_{\sigma \lambda} z_\lambda^{(s)}}{\sum m_{\sigma \lambda} z_\lambda^{(s)}} \]

Positive-definite if the potential is a minimum at equilibrium:

\[ V(q_1, \ldots, q_n) = V(q_1^*, \ldots, q_n^*) + \sum \eta_\sigma \left( \frac{\partial V}{\partial q_\sigma} \right) q_\sigma + \frac{1}{2} \sum \eta_\sigma \eta_\lambda \left( \frac{\partial^2 V}{\partial q_\sigma \partial q_\lambda} \right) q_\sigma q_\lambda \]

Positive-definite, since \( m \) is the mass matrix:

\[ T = \frac{1}{2} \sum \sum m_{\sigma \lambda} \dot{\eta}_\sigma \dot{\eta}_\lambda \]

\[ \sum_{\lambda=1}^{n} (v_{\sigma \lambda} - \omega_s^2 m_{\sigma \lambda}) z_\lambda^{(s)} = 0 \quad \sigma = 1, \ldots, n \]

N-1 ratios of components \( z_s^{(s)}/z_n^{(s)} \) are real, there can be only an overall phase:

\[ z_\sigma^{(s)} = e^{i\phi_s} \rho_\sigma^{(s)} \quad \sigma = 1, \ldots, n \]

One component can be chosen arbitrarily:

n-1 ratios of components are real, there can be only an overall phase.
Form of the general solution (continued):

**s-th eigenvalue:**

\[
\sum_{\lambda} (v_{\sigma \lambda} - \omega_s^2 m_{\sigma \lambda}) z^{(s)}_{\lambda} = 0 \quad \sigma = 1, \ldots, n
\]

\[
z^{(s)}_{\sigma} = e^{i\phi_{\sigma}} \rho^{(s)}_{\sigma} \quad \sigma = 1, \ldots, n
\]

**t-th eigenvalue:**

\[
\sum_{\lambda} v_{\lambda \sigma} \rho^{(t)}_{\lambda} = \omega_t^2 \sum_{\lambda} m_{\lambda \sigma} \rho^{(t)}_{\lambda} \quad \sigma = 1, \ldots, n
\]

\[
\sum_{\sigma} \rho^{(t)}_{\sigma}
\]

\[
\sum_{\sigma} \rho^{(s)}_{\sigma}
\]

\[
(m_{\sigma \lambda} = m_{\lambda \sigma} = m^*_{\sigma \lambda} = m^*_{\lambda \sigma})
\]

\[
v_{\sigma \lambda} = v_{\lambda \sigma} = v^*_{\sigma \lambda} = v^*_{\lambda \sigma}
\]

**For non-degenerate eigenvalues:**

we can normalize the eigenvectors according to

\[
\sum_{\lambda} \sum_{\sigma} \rho^{(t)}_{\sigma} m_{\sigma \lambda} \rho^{(s)}_{\lambda} = 0 \quad s \neq t
\]

and write the general solution as:

\[
z^{(s)}_{\sigma} = C^{(s)} e^{i\phi_{\sigma}} \rho^{(s)}_{\sigma} \quad \sigma = 1, \ldots, n
\]

**for degenerate eigenvalues we can use Gram-Schmidt orthogonalization procedure to get**
Putting things together:

General solution:

2n solutions with 4n independent constants

n solutions with 2n independent constants → the most general solution!