Representations of Lorentz Group

We defined a unitary operator that implemented a Lorentz transformation on a scalar field:

\[ U(\Lambda)^{-1} \phi(x) U(\Lambda) = \phi(\Lambda^{-1} x) \]

and then a derivative transformed as:

\[ U(\Lambda)^{-1} \partial^\mu \phi(x) U(\Lambda) = \Lambda^\mu_\rho \partial^\rho \phi(\Lambda^{-1} x) \]

it suggests, we could define a vector field that would transform as:

\[ U(\Lambda)^{-1} \Lambda^\mu(x) U(\Lambda) = \Lambda^\mu_\rho A^\rho(\Lambda^{-1} x) \]

and a tensor field \( B^{\mu\nu}(x) \) that would transform as:

\[ U(\Lambda)^{-1} B^{\mu\nu}(x) U(\Lambda) = \Lambda^\mu_\rho \Lambda^\nu_\sigma B^{\rho\sigma}(\Lambda^{-1} x) \]

for symmetric \( B^{\mu\nu}(x) = B^{\nu\mu}(x) \) and antisymmetric \( B^{\mu\nu}(x) = -B^{\nu\mu}(x) \)
tensors, the symmetry is preserved by Lorentz transformations.

In addition, the trace \( T(x) \equiv g_{\mu\nu} B^{\mu\nu}(x) \) transforms as a scalar:

\[ g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = g_{\rho\sigma} \]

Thus a general tensor field can be written as:

\[ B^{\mu\nu}(x) = \Lambda^{\mu\nu}(x) + S^{\mu\nu}(x) + \frac{1}{4} g^{\mu\nu} T(x) \]

where different parts do not mix with each other under LT!

How do we find the smallest (irreducible) representations of the Lorentz group for a field with \( n \) vector indices?

Let's start with a field carrying a generic Lorentz index:

\[ U(\Lambda)^{-1} \phi_A(x) U(\Lambda) = L_A^B(\Lambda) \phi_B(\Lambda^{-1} x) \]

we say these matrices form a representation of the Lorentz group.

For an infinitesimal transformation we had:

\[ U(1 + \delta \omega) = I + \frac{1}{2} \delta \omega^{\mu\nu} M^{\mu\nu} \]

where the generators of the Lorentz group satisfied:

\[ [M^{\mu\nu}, M^{\rho\sigma}] = i \left( g^{\mu\rho} M^{\nu\sigma} - (\mu \leftrightarrow \nu) - (\rho \leftrightarrow \sigma) \right) \]

Lie algebra of the Lorentz group.

or in components (angular momentum and boost),

\[ J_i \equiv \frac{1}{2} \epsilon_{ijk} M^{jk} \]

we have found:

\[ [J_i, J_j] = i \hbar \epsilon_{ijk} J_k \]

\[ [K_i, J_j] = i \hbar \epsilon_{ijk} K_k \]

\[ [K_i, K_j] = -i \hbar \epsilon_{ijk} J_k \]
In a similar way, for an infinitesimal transformation we also define:

\[ U(1 + \delta \omega) = I + \frac{1}{2} \delta \omega_{\mu \nu} M^{\mu \nu} \]

not necessarily hermitian

\[ L_A^B (1 + \delta \omega) = \delta_A^B + \frac{1}{2} \delta \omega_{\mu \nu} \left( S^{\mu \nu} \right)_A^B \]

and we find:

\[ U(\Lambda)^{-1} \varphi_A(x) U(\Lambda) = L_A^B(\Lambda) \varphi_B(\Lambda^{-1} x) \]

comparing linear terms in \( \delta \omega_{\mu \nu} \)

\[ [\varphi_A(x), M^{\mu \nu}] = L^{\mu \nu} \varphi_A(x) + (S^{\mu \nu})_A^B \varphi_B(x) \]

\[ L^{\mu \nu} = \frac{i}{2} (\partial^\mu \partial^\nu - \partial^\nu \partial^\mu) \]

also it is possible to show that \( L^{\mu \nu} \) and \( (S^{\mu \nu})_A^B \) obey the same commutation relations as the generators

\[ [M^{\mu \nu}, M^{\rho \sigma}] = i \left( g^{\mu \rho} M^{\nu \sigma} - (\mu \leftrightarrow \nu) \right) - (\rho \leftrightarrow \sigma) \]

How do we find all possible sets of matrices that satisfy \( \downarrow \)?

\[ [M^{\mu \nu}, M^{\rho \sigma}] = i \left( g^{\mu \rho} M^{\nu \sigma} - (\mu \leftrightarrow \nu) \right) - (\rho \leftrightarrow \sigma) \]

the first one is just the usual set of commutation relations for angular momentum in QM:

for given \( j \) \((0, 1/2, 1, \ldots)\) we can find three \((2j+1)\times(2j+1)\) hermitian matrices \( J_1, J_2 \) and \( J_3 \) that satisfy the commutation relations and the eigenvalues of \( J_3 \) are \(-j, -j+1, \ldots, +j\).

such matrices constitute all of the inequivalent, irreducible representations of the Lie algebra of \( SO(3) \) equivalent to the Lie algebra of \( SU(2) \)

not related by a unitary transformation

\[ [J_i, J_j] = i \hbar \varepsilon_{ijk} J_k \]

\[ [J_i, K_j] = i \hbar \varepsilon_{ijk} K_k \]

\[ [K_i, K_j] = -i \hbar \varepsilon_{ijk} J_k \]

\[ [N_i, N_j] = i \varepsilon_{ijk} N_k \]

\[ [N_i^+, N_j^+] = i \varepsilon_{ijk} N_k^+ \]

\[ [N_i, N_j^+] = 0 \]

The Lie algebra of the Lorentz group splits into two different \( SU(2) \) Lie algebras that are related by hermitian conjugation!

A representation of the Lie algebra of the Lorentz group can be specified by two integers or half-integers:

\( (2n+1, 2n'+1) \)

there are \((2n+1)(2n'+1)\) different components of a representation they can be labeled by their angular momentum representations:

since \( J_i = N_i + N_i^+ \) for given \( n \) and \( n' \) the allowed values of \( j \) are

\[ |n-n'|, |n-n'|+1, \ldots, n+n' \]

(the standard way to add angular momenta, each value appears exactly once)

The simplest representations of the Lie algebra of the Lorentz group are:

\( (2n+1, 2n'+1) \)

\((1, 1) = \text{scalar or singlet}\)

\((2, 1) = \text{left-handed spinor}\)

\((1, 2) = \text{right-handed spinor}\)

\((2, 2) = \text{vector}\)

\( j = 0 \) and \( 1 \)
Left- and Right-handed spinor fields

Let’s start with a left-handed spinor field (left-handed Weyl field) $\psi_a(x)$:

under Lorentz transformation we have:

$$U(\Lambda)^{-1}\psi_a(x)U(\Lambda) = L_a^b(\Lambda)\psi_b(\Lambda^{-1}x)$$

left-handed spinor index

matrices in the (2,1) representation, that satisfy the group composition rule:

$$L_a^b(\Lambda^\prime)L_b^c(\Lambda) = L_a^c(\Lambda^\prime\Lambda)$$

For an infinitesimal transformation we have:

$$L_a^b(1+\delta) = \delta_a^b + \frac{i}{2} \delta \omega_{\mu\nu}(S_L^\mu\nu)_a^b$$

$$[S_L^\mu\nu, S_L^\rho\sigma] = i(g^\rho\sigma S_L^\mu\nu - (\mu\nu\rho\sigma) - (\rho\sigma\mu\nu))$$

Using

$$U(1+\delta) = I + \frac{i}{2} \delta \omega_{\mu\nu} M^{\mu\nu}$$

we get

$$[\psi_a(x), M^{\mu\nu}] = \mathcal{L}^{\mu\nu}\psi_a(x) + (S_L^\mu\nu)_a^b \psi_b(x)$$

$$\mathcal{L}^{\mu\nu} \equiv \frac{1}{2}(x^\mu \partial^\nu - x^\nu \partial^\mu)$$

present also for a scalar field to simplify the formulas, we can evaluate everything at space-time origin, $x^\mu = 0$

and since $M^{ij} = \epsilon^{ijk} J_k$, we have:

$$\epsilon^{ijk}[\psi_a(0), J_k] = (S_L^{ij})_a^b \psi_b(0)$$

so that for $i=1$ and $j=2$:

$$(S_L^{12})_a^b = \frac{1}{2} \epsilon^{123} \sigma_k = \frac{1}{2} \sigma_3$$

$$(S_L^{12})_a^1 = +1, (S_L^{22})_a^2 = -\frac{1}{2}$$

$$(S_L^{21}) = (S_L^{12})_a^1 = 0$$

standard convention

Let’s consider now a hermitian conjugate of a left-handed spinor field $\psi_a(x)$ (a hermitian conjugate of a (2,1) field should be a field in the (1,2) representation) = right-handed spinor field (right-handed Weyl field)

$$[\psi_a(x)]^\dagger = \psi_\dagger(x)$$

we use dotted indices to distinguish (2,1) from (1,2)!

under Lorentz transformation we have:

$$U(\Lambda)^{-1}\psi_\dagger(x)U(\Lambda) = R_a^b(\Lambda)\psi_\dagger(\Lambda^{-1}x)$$

matrices in the (1,2) representation, that satisfy the group composition rule:

$$R_a^b(\Lambda^\prime)R_b^c(\Lambda) = R_a^c(\Lambda^\prime\Lambda)$$

For an infinitesimal transformation we have:

$$R_a^b(1+\delta) = \delta_a^b + \frac{i}{2} \delta \omega_{\mu\nu}(S_R^{\mu\nu})_a^b$$

$$[S_R^{\mu\nu}, S_R^{\rho\sigma}] = i(g^\rho\sigma S_R^{\mu\nu} - (\mu\nu\rho\sigma) - (\rho\sigma\mu\nu))$$
in the same way as for the left-handed field we find:

\[ [\psi^\dagger_a(0), M^{\mu\nu}] = (S^{\mu\nu}_R)_{a}^{b} \psi^\dagger_b(0) \]

taking the hermitian conjugate,

\[ [M^{\mu\nu}, \psi_a(0)] = [(S^{\mu\nu}_R)_{a}^{b}]^* \psi_b(0) \]

we find:

\[ (S^{\mu\nu}_R)_{a}^{b} = -[(S^{\mu\nu}_R)_{a}^{b}]^* \]

Let's consider now a field that carries two (2,1) indices.

Under Lorentz transformation we have:

\[ U(\Lambda)^{-1}C_{ab}(x)U(\Lambda) = L_a^c(\Lambda)L_b^d(\Lambda)C_{cd}(\Lambda^{-1}x) \]

Can we group 4 components of C into smaller sets that do not mix under Lorentz transformation?

Recall from QM that two spin 1/2 particles can be in a state of total spin 0 or 1:

\[ 2 \otimes 2 = 1_A \oplus 3_S \]

1 antisymmetric spin 0 state \quad 3 symmetric spin 1 states

Thus for the Lorentz group we have:

\[ (2, 1) \otimes (2, 1) = (1, 1)_A \oplus (3, 1)_S \]

and we should be able to write:

\[ C_{ab}(x) = \varepsilon_{ab}D(x) + G_{ab}(x) \]

\[ \varepsilon_{ab} = -\varepsilon_{ba} \]

\[ G_{ab}(x) = G_{ba}(x) \]

\[ \varepsilon_{ab} = -\varepsilon_{ba} \]

\[ \varepsilon_{12} = \varepsilon_{21} = +1, \quad \varepsilon_{21} = \varepsilon_{12} = -1 \]

\[ \varepsilon_{ab}\varepsilon^{bc} = \delta_a^c, \quad \varepsilon_{ab}\varepsilon_{bc} = \delta^a_c \]

to raise and lower left-handed spinor indices:

\[ \psi^a(x) \equiv \varepsilon^{ab}\psi_b(x) \]

\[ \varepsilon_{ab}\varepsilon^{bc} = \delta_a^c, \quad \varepsilon_{ab}\varepsilon_{bc} = \delta^a_c \]

\[ \psi^a(x) \equiv \varepsilon^{ab}\psi_b(x) \]

We also have:

\[ \psi_a = \varepsilon_{ab}\psi^b = \varepsilon_{ab}\varepsilon^{bc}\psi_c = \delta_a^c\psi_c \]

we have to be careful with the minus sign, e.g.:

\[ \psi^a = \varepsilon^{ab}\psi_b = -\varepsilon^{ba}\psi_b = -\psi_b\varepsilon^{ba} = \psi_b\varepsilon^{ab} \]

or when contracting indices:

\[ \psi^a\chi_a = \varepsilon^{ab}\psi_b\chi_a = -\varepsilon^{ba}\psi_b\chi_a = -\psi_b\chi^b \]

Exactly the same discussion applies to two (1,2) indices:

\[ (1, 2) \otimes (1, 2) = (1, 1)_A \oplus (1, 3)_S \]

with \( \varepsilon_{\dot{a}\dot{b}} \) defined in the same way as \( \varepsilon_{ab} \):

\[ \varepsilon_{\dot{a}\dot{b}} = -\varepsilon_{\dot{b}\dot{a}} \]
Finally, let’s consider a field that carries one undotted and one dotted index; it is in the (2,2) representation (vector):

\[ A_{\alpha \dot{\alpha}}(x) = \sigma^\mu_{\alpha \dot{\alpha}} A^\mu(x) \]

A consistent choice with what we have already set for \( S^\mu_{\mathcal{L}} \) and \( S^\mu_{\mathcal{R}} \) is:

\[ \sigma^\mu_{\alpha \dot{\alpha}} = (I, \vec{\sigma}) \]

In general, whenever the product of a set of representations includes the singlet, there is a corresponding invariant symbol, e.g. the existence of \( g_{\mu \nu} = g_{\nu \mu} \) follows from

\[ (2,2) \otimes (2,2) = (1,1)_S \oplus (1,3)_A \oplus (3,1)_A \oplus (3,3)_S \]

another invariant symbol we will use is completely antisymmetric Levi-Civita symbol:

\[ \varepsilon^{\mu \nu \rho \sigma} \]

\[ \varepsilon^{0123} = +1 \]

\( \Lambda^\mu_{\alpha} \Lambda^\nu_{\beta} \Lambda^\gamma_{\gamma} \Lambda^\delta_{\delta} \varepsilon^{\alpha \beta \gamma \delta} \) is antisymmetric on exchange of any two of its uncontracted indices, and therefore must be proportional to \( \varepsilon^{\mu \nu \rho \sigma} \), the constant of proportionality is \( \det \Lambda \) which is +1 for proper Lorentz transformations.

Comparing the formula for a general field with two vector indices

\[ B^{\mu \nu}(x) = A^{\mu \nu}(x) + S^\nu_{\mathcal{L}}(x) + \frac{1}{4} g^{\mu \nu} T(x) \]

with

\[ (2,2) \otimes (2,2) = (1,1)_S \oplus (1,3)_A \oplus (3,1)_A \oplus (3,3)_S \]

we see that \( A \) is not irreducible and, since \( (3,1) \) corresponds to a symmetric part of undotted indices,

\[ 2 \otimes 2 = 1_A \oplus 3_S \]

\[ C_{a\dot{b}}(x) = \varepsilon_{a\dot{b}} D(x) + G_{a\dot{b}}(x) \]

we should be able to write it in terms of \( G \) and its hermitian conjugate.

see Srednicki

Fun with spinor indices

based on S-35

invariant symbol for raising and lowering spinor indices:

\[ \psi^a(x) \equiv \varepsilon^{a\dot{b}} \psi_\dot{b}(x) \]

\[ \varepsilon^{12} = \varepsilon^{1\dot{2}} = \varepsilon^{21} = \varepsilon^{2\dot{1}} = +1 \]\n
\[ \varepsilon^{21} = \varepsilon^{2\dot{1}} = \varepsilon^{12} = \varepsilon^{1\dot{2}} = -1 \]

\[ \varepsilon^{a\dot{b}} = -\varepsilon^{\dot{b}a} = i \sigma_2 \]

another invariant symbol:

\[ \sigma^\mu_{a\dot{a}} = (I, \vec{\sigma}) \]

\[ A_{a\dot{a}}(x) = \sigma^\mu_{a\dot{a}} A^\mu(x) \]

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

Simple identities:

\[ \sigma^\mu_{a\dot{a}} \sigma^a_{\mu \dot{b}} = -2 \varepsilon_{a\dot{b}} \varepsilon_{\dot{a}b} \]

\[ \varepsilon^{a\dot{b}} \varepsilon^{a\dot{b}} \sigma^\mu_{a\dot{a}} \sigma^\nu_{b\dot{b}} = -2 g^{\mu \nu} \]

proportionality constants from direct calculation
What can we learn about the generator matrices \((S_L^{\mu\nu})_a^b\) from invariant symbols?

\[ \varepsilon_{ab} = L(\Lambda)_a^c L(\Lambda)_b^d \varepsilon_{cd} : \]

for an infinitesimal transformation we had:

\[ L_a^b (1 + \delta \omega) = \delta_a^b + \frac{i}{2} \delta \omega_{\mu \nu} (S_L^{\mu\nu})_a^b \]

and we find:

\[ \varepsilon_{ab} = \varepsilon_{ab} + \frac{i}{2} \delta \omega_{\mu \nu} \left[ (S_L^{\mu\nu})_a^c \varepsilon_{cb} + (S_L^{\mu\nu})_b^c \varepsilon_{ad} \right] + O(\delta \omega^2) \]

\[ = \varepsilon_{ab} + \frac{i}{2} \delta \omega_{\mu \nu} \left[ -(S_L^{\mu\nu})_{ab} + (S_L^{\mu\nu})_{ba} \right] + O(\delta \omega^2) . \]

Similarly:

\[ (S_L^{\mu\nu})_{ab} = (S_L^{\mu\nu})_{ba} \]

Convection:

missing pair of contracted indices is understood to be written as:

\[ \sigma_{a^a b^b} = \Lambda_\alpha^a \Lambda_\beta^b R(\Lambda)^b_a \sigma_{\alpha\beta} \]

for infinitesimal transformations we had:

\[ \Lambda_\rho^a = \delta_\rho^a + \frac{i}{2} \delta \omega_{\mu \nu} (S_L^{\mu\nu})_a^b, \]

isotating linear terms in \( \delta \omega_{\mu \nu} \) we have:

\[ (g^{\mu \nu} \delta_{\mu}^\rho - g^{\nu \rho} \delta_{\nu}^\mu) \sigma_{a^a}^\rho + i (S_L^{\mu\nu})_a^b \sigma_{b^b}^\rho + i (S_L^{\mu\nu})_b^a \sigma_{a^a}^\rho = 0 \]

multiplying by \( \sigma_{\rho \sigma \rho} \) we have:

\[ \sigma_{c^c d^d}^{\mu \nu} - \sigma_{c^c d^d}^{\nu \mu} + i (S_L^{\mu\nu})_a^b \sigma_{b^b \sigma d^d}^\rho + i (S_L^{\mu\nu})_b^a \sigma_{a^a \sigma c^c}^\rho = 0 \]

\[ \sigma_{a^a d^d}^{\mu \nu} - \sigma_{a^a d^d}^{\nu \mu} + 2i (S_L^{\mu\nu})_{ac} \sigma_{d^d e^e c^c} + 2i (S_L^{\mu\nu})_{bc} \sigma_{a^a e^e d^d} = 0 \]

thus, for left-handed Weyl fields we have:

\[ \chi \psi = \chi^2 \psi_a \] and \[ \chi^\dagger \psi^\dagger = \chi^\dagger \psi^{\dagger a} \]

spin 1/2 particles are fermions that anticommute:

the spin-statistics theorem (later)

and we find:

\[ \chi \psi = \chi \psi_a = -\psi_a \chi = \psi_a \chi = \psi \chi \]

\[ a^a = -a^a \]
spin 1/2 particles are fermions that anticommute:

\[ \chi_a(x) \psi_b(y) = -\psi_b(y) \chi_a(x) \]

and we find:

\[ \chi^\dagger \psi = \chi^a \psi_a = -\psi_a \chi^a = \psi^a \chi_a = \psi \chi \]

for hermitian conjugate we find:

\[ (\chi^\dagger)^\dagger = (\chi^a)^\dagger \]

as expected if we ignored indices

and similarly:

\[ \psi^\dagger \chi^\dagger = \chi^\dagger \psi^\dagger \]

we will write a right-handed field always with a dagger!

Let's look at something more complicated:

\[ \psi^\dagger \tilde{\sigma}^\mu \chi = \psi^\dagger \tilde{\sigma}^{\mu \nu} \chi^\nu \]

it behaves like a vector field under Lorentz transformations:

\[ U(\Lambda)^{-1} [\psi^\dagger \tilde{\sigma}^\mu \chi] U(\Lambda) = \Lambda^\mu \nu [\psi^\dagger \tilde{\sigma}^\nu \chi] \]

the hermitian conjugate is:

\[ [\psi^\dagger \tilde{\sigma}^\mu \chi]^\dagger = [\psi^\dagger \tilde{\sigma}^{\mu \nu} \chi^\nu]^\dagger \]

\[ = \chi^\dagger (\tilde{\sigma}^{\mu \nu})^* \psi_a \]

\[ = \chi^\dagger \tilde{\sigma}^\mu \chi \]

\[ \tilde{\sigma}^\mu = (I, -\tilde{\sigma}) \text{ is hermitian} \]