Lorentz invariance

Lorentz transformation (linear, homogeneous change of coordinates):
\[ \vec{x}^\mu = \Lambda^\mu_\nu x^\nu \]
that preserves the interval \( x^2 \):
\[ x^2 \equiv x^\mu x_\mu = g_{\mu\nu}x^\mu x^\nu = x^2 - c^2 t^2. \]

All Lorentz transformations form a group:
- product of 2 LT is another LT
- identity transformation: \( \Lambda^\mu_\nu = \delta^\mu_\nu \)
- inverse: \( (\Lambda^{-1})^\mu_\nu = \Lambda^\nu_\rho \)

\[ g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = g_{\rho\sigma} \]
for inverse can be used to prove:
\[ g^{\mu\nu} \Lambda^\rho_\mu \Lambda^\sigma_\nu = g^{\rho\sigma} \]

Infinitesimal Lorentz transformation:
\[ \Lambda^\mu_\nu = \delta^\mu_\nu + \delta \omega^\mu_\nu \]

\[ g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = g_{\rho\sigma} \]
\[ \delta \omega_{\rho\sigma} = -\delta \omega_{\sigma\rho} \]

thus there are 6 independent ILTs: 3 rotations and 3 boosts

not all LT can be obtained by compounding ILTs!
\[ (\Lambda^{-1})^\rho_\nu = \Lambda^\nu_\rho \quad (\text{det } \Lambda)^{-1} = \text{det } \Lambda \quad \text{det } \Lambda = \pm 1 \]

proper LTs form a subgroup of Lorentz group; ILTs are proper!

Another subgroup - orthochronous LTs, \( \Lambda^0_0 \geq +1 \)
\[ g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = g_{\rho\sigma} \]
\[ (\Lambda^0_0)^2 - \Lambda^i_0 \Lambda^i_0 = 1 \]
\[ \Lambda^0_0 \leq -1 \]

ILT are orthochronous!
When we say theory is Lorentz invariant we mean it is invariant under proper orthochronous subgroup only (those that can be obtained by compounding ILTs)

Transformations that take us out of proper orthochronous subgroup are parity and time reversal:

\[
P_{\mu \nu} = (P^{-1})_{\mu \nu} = \begin{pmatrix} +1 & -1 \\ -1 & -1 \end{pmatrix} \quad \text{orthochronous but improper}
\]

\[
T_{\mu \nu} = (T^{-1})_{\mu \nu} = \begin{pmatrix} -1 & +1 \\ +1 & +1 \end{pmatrix} \quad \text{nonorthochronous and improper}
\]

A quantum field theory doesn’t have to be invariant under P or T.

How do operators and quantum fields transform?

Lorentz transformation (proper, orthochronous) is represented by a unitary operator \( U(\Lambda) \) that must obey the composition rule:

\[
U(\Lambda' \Lambda) = U(\Lambda')U(\Lambda)
\]

Infinitesimal transformation can be written as:

\[
U(1 + \delta \omega) = I + \frac{i}{2\hbar} \delta \omega_{\mu \nu} M^{\mu \nu}
\]

\( M^{\mu \nu} \) are hermitian operators = generators of the Lorentz group

From

\[
U(\Lambda)^{-1}U(\Lambda')U(\Lambda) = U(\Lambda^{-1} \Lambda') \quad \text{using} \quad \Lambda' = 1 + \delta \omega'
\]

and expanding both sides, keeping only linear terms in \( \delta \omega \) we get:

\[
\delta \omega_{\mu \nu} U(\Lambda)^{-1} M^{\mu \nu} U(\Lambda) = \delta \omega_{\mu \nu} \Lambda^\rho_\mu \Lambda^\nu_\sigma M^{\rho \sigma}
\]

since \( \delta \omega_{\mu \nu} \) are arbitrary

\[
U(\Lambda)^{-1} M^{\mu \nu} U(\Lambda) = \Lambda^\rho_\mu \Lambda^\nu_\sigma M^{\rho \sigma}
\]

General rule: each vector index undergoes its own Lorentz transformation!
\[ U(\Lambda)^{-1} M^{\mu\nu} U(\Lambda) = \Lambda^\mu_\rho \Lambda^\nu_\sigma M^{\rho\sigma} \]

using \( \Lambda = 1 + \delta \omega \) and expanding to linear order in \( \delta \omega_{\mu\nu} \) we get:

\[ [M^{\mu\nu}, M^{\rho\sigma}] = i\hbar \left( g^{\mu\rho} M^{\nu\sigma} - (\mu \leftrightarrow \nu) \right) - \rho \leftrightarrow \sigma \]

These comm. relations specify the Lie algebra of the Lorentz group.

We can identify components of the angular momentum and boost operators:

\[ J_i \equiv \frac{1}{2} \varepsilon_{ijk} M^{jk} \]

\[ K_i \equiv M^{i0} \]

and find:

\[ [J_i, J_j] = i\hbar \varepsilon_{ijk} J_k , \]

\[ [J_i, K_j] = i\hbar \varepsilon_{ijk} K_k , \]

\[ [K_i, K_j] = -i\hbar \varepsilon_{ijk} J_k \]

in a similar way for the energy-momentum four vector \( P^\mu = (H/c, P^i) \) we find:

\[ U(\Lambda)^{-1} P^\mu U(\Lambda) = \Lambda^\mu_\nu P^\nu \]

using \( \Lambda = 1 + \delta \omega \) and expanding to linear order in \( \delta \omega_{\mu\nu} \) we get:

\[ [P^\mu, M^{\rho\sigma}] = i\hbar \left( g^{\mu\sigma} P^\rho - (\rho \leftrightarrow \sigma) \right) \]

or in components:

\[ [J_i, H] = 0 , \]

\[ [J_i, P_j] = i\hbar \varepsilon_{ijk} P_k , \]

\[ [K_i, H] = i\hbar P_i , \]

\[ [K_i, P_j] = i\hbar \delta_{ij} H , \]

in addition:

\[ [P_i, P_j] = 0 , \]

\[ [P_i, H] = 0 . \]

Comm. relations for \( J, K, P, H \) form the Lie algebra of the Poincare group.
Finally, let’s look at transformation of a quantum scalar field:

Recall time evolution in Heisenberg picture:

\[ e^{+iHt/\hbar} \varphi(x, 0) e^{-iHt/\hbar} = \varphi(x, t) \]

this is generalized to:

\[ e^{-iPx/\hbar} \varphi(0) e^{+iPx/\hbar} = \varphi(x) \]

\[ P_x = P^\mu x_\mu = \mathbf{P} \cdot \mathbf{x} - Ht \]

\( x \) is just a label

we can write the same formula for \( x-a \):

\[ e^{+iPa/\hbar} e^{-iPx/\hbar} \phi(0) e^{+iPx/\hbar} e^{-iPa/\hbar} = \phi(x-a) \]

we define space-time translation operator:

\[ T(a) \equiv \exp(-iP^\mu a_\mu/\hbar) \]

\[ T(\delta a) = I - \frac{i}{\hbar} \delta a_\mu P^\mu \]

and obtain:

\[ T(a)^{-1} \varphi(x) T(a) = \varphi(x-a) \]

Similarly:

\[ U(\Lambda)^{-1} \varphi(x) U(\Lambda) = \varphi(\Lambda^{-1}x) \]

Derivatives carry vector indices:

\[ U(\Lambda)^{-1} \partial^\mu \varphi(x) U(\Lambda) = \Lambda^{\mu\rho} \partial^\rho \varphi(\Lambda^{-1}x) \]

\[ \bar{x} = \Lambda^{-1}x \]

\[ U(\Lambda)^{-1} \partial^2 \varphi(x) U(\Lambda) = \bar{\partial}^2 \varphi(\Lambda^{-1}x) \]

\[ (-\bar{\partial}^2 + m^2/\hbar^2 c^2) \varphi = 0 \]

is Lorentz invariant
Canonical quantization of scalar fields

Based on S-3 Hamiltonian for free nonrelativistic particles:

\[ H = \int d^3x \ a^\dagger(x) \left( -\frac{1}{2m} \nabla^2 \right) a(x) \]

\( \hbar = 1 \)

Furier transform:

\[ a(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \ e^{ip \cdot x} \tilde{a}(p) \]

we get:

\[ H = \int d^3p \ \frac{1}{2m} p^2 \ a^\dagger(p) \tilde{a}(p) \]

can go back to \( x \) using:

\[ \tilde{a}(p) = \int \frac{d^3x}{(2\pi)^{3/2}} \ e^{-ip \cdot x} a(x) \]

28

Canonical quantization of scalar fields

(Anti)commutation relations:

\[ [a(x), a(x')] = 0 \]

\[ [a^\dagger(x), a^\dagger(x')] = 0 \]

\[ [a(x), a^\dagger(x')] = \delta^3(x - x') \]

\[ [\tilde{a}(p), \tilde{a}(p')] = 0 \]

\[ [\tilde{a}^\dagger(p), \tilde{a}^\dagger(p')] = 0 \]

\[ [\tilde{a}(p), \tilde{a}^\dagger(p')] = \delta^3(p - p') \]

\[ [A, B] = AB + BA \]

Vacuum is annihilated by \( \tilde{a}(p) \): \n
\[ \tilde{a}(p)|0\rangle = 0 \]

\( \tilde{a}^\dagger(p) \ |0\rangle \) is a state of momentum \( p \), eigenstate of \( H \) with \( E(p) = \frac{1}{2m} p^2 \).

\( \tilde{a}^\dagger(p_1) \ldots \tilde{a}^\dagger(p_n) |0\rangle \) is eigenstate of \( H \) with energy eigenvalue:

\[ E(p_1) + \ldots + E(p_n) \]
Relativistic generalization

Hamiltonian for free relativistic particles:

\[ E(p) = +(p^2c^2 + m^2c^4)^{1/2} \]

\[ H = \int d^3p \left( p^2c^2 + m^2c^4 \right)^{1/2} \tilde{a}^\dagger(p) \tilde{a}(p) \]

spin zero, but can be either bosons or fermions

Is this theory Lorentz invariant?

Let’s prove it from a different direction, direction that we will use for any quantum field theory from now:

- start from a Lorentz invariant lagrangian or action
- derive equation of motion (for scalar fields it is K.-G. equation)
- find solutions of equation of motion
- show the Hamiltonian is the same as the one above

A theory is described by an action:

\[ S = \int dt \ L \]

where \( L \) is the lagrangian.

Equations of motion should be local, and so

\[ L = \int d^3x \ L \]

where \( L \) is the lagrangian density.

Thus:

\[ S = \int d^4x \ L \]

\( d^4x \) is Lorentz invariant: \( \bar{x}^\mu = \Lambda^\mu_\nu x^\nu \rightarrow d^4x = |\text{det} \Lambda| \ d^4x = d^4x \)

For the action to be invariant we need: \( \text{det} \Lambda = \pm 1 \)

\[ \mathcal{L}(x) = \bar{\mathcal{L}}(\bar{x}) \rightarrow \bar{S} = \int d^4\bar{x} \bar{\mathcal{L}}(\bar{x}) = \int d^4x \mathcal{L}(x) = S \]

the lagrangian density must be a Lorentz scalar!
Any polynomial of a scalar field is a Lorentz scalar and so are products of derivatives with all indices contracted.

Let’s consider:

\[ \mathcal{L} = -\frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} m^2 \varphi^2 + \Omega_0 \]

and let’s find the equation of motion, Euler-Lagrange equation:

(we find eq. of motion from variation of an action: making an infinitesimal variation \( \delta \varphi(x) \) in \( \varphi(x) \) and requiring the variation of the action to vanish)

\[ 0 = \delta S = \int d^4x \left[ -\frac{1}{2} \partial^\mu \delta \varphi \partial_\mu \varphi - \frac{1}{2} \partial^\mu \varphi \partial_\mu \delta \varphi - m^2 \varphi \delta \varphi \right] \]

\[ = \int d^4x \left[ +\partial^\mu \partial_\mu \varphi - m^2 \varphi \right] \delta \varphi. \]

\( \delta \varphi(x) \) is arbitrary function of \( x \) and so the equation of motion is

Klein-Gordon equation

Solutions of the Klein-Gordon equation:

\[ (-\partial^2 + m^2) \varphi = 0 \]

one classical solution is a plane wave:

\[ \exp(i \mathbf{k} \cdot \mathbf{x} \pm i \omega t) \]

\( \mathbf{k} \) is arbitrary real wave vector and \( \omega = + (k^2 + m^2)^{1/2} \)

The general classical solution of K-G equation:

\[ \varphi(x, t) = \int \frac{d^3k}{f(k)} \left[ a(k)e^{i \mathbf{k} \cdot \mathbf{x} - i \omega t} + b(k)e^{i \mathbf{k} \cdot \mathbf{x} + i \omega t} \right] \]

where \( a(k) \) and \( b(k) \) are arbitrary functions of \( \mathbf{k} \), and

\( f(k) \) is a function of \( |\mathbf{k}| \) (introduced for later convenience)

if we tried to interpret \( \varphi(x) \) as a quantum wave function, the second term would represent contributions with negative energy to the wave function!
\[ \varphi(x, t) = \int \frac{d^3k}{f(k)} \left[ a(k)e^{ik \cdot x - i\omega t} + b(k)e^{ik \cdot x + i\omega t} \right] \]

real solutions: \[ \varphi^*(x) = \varphi(x) \quad \Rightarrow \quad b^*(-k) = a(k) \]

\[ \varphi^*(x, t) = \int \frac{d^3k}{f(k)} \left[ a^*(k)e^{-ik \cdot x + i\omega t} + b^*(k)e^{-ik \cdot x - i\omega t} \right] \]

\[ = \int \frac{d^3k}{f(k)} \left[ a^*(k)e^{-ik \cdot x + i\omega t} + b^*(-k)e^{ik \cdot x - i\omega t} \right] \]

thus we get:

\[ \varphi(x, t) = \int \frac{d^3k}{f(k)} \left[ a(k)e^{ik \cdot x - i\omega t} + a^*(-k)e^{ik \cdot x + i\omega t} \right] \]

\[ k \cdot x = k \cdot x - \omega t \]
\[ \mu\nu = g_{\mu\nu}k^\mu x^\nu \]
\[ k^\mu = (\omega, k) \]
\[ x^\mu = (t, x) \]

\[ k^2 = k^\mu k_\mu = k^2 - \omega^2 = -m^2 \quad \text{(such a } k^\mu \text{ is said to be on the mass shell)} \]

Finally let’s choose \( f(k) \) so that \( \frac{d^3k}{f(k)} \) is Lorentz invariant:

\[ d^4k \delta(k^2 + m^2) \theta(k^0) \]

manifestly invariant under orthochronous Lorentz transformations on the other hand

\[ \int_{-\infty}^{+\infty} dk^0 \delta(k^2 + m^2) \theta(k^0) = \frac{1}{2\omega} \]

\[ \int_{-\infty}^{+\infty} dx \delta(g(x)) = \sum_{i} \frac{1}{|g'(x_i)|} \]

sum over zeros of \( g \), in our case the only zero is \( k^0 = \omega \)

for any \( f(k) \propto \omega \) the differential \( \frac{d^3k}{f(k)} \) is Lorentz invariant

it is convenient to take \( f(k) = (2\pi)^3 2\omega \) for which

the Lorentz invariant differential is:

\[ \tilde{d}^3k \equiv \frac{d^3k}{(2\pi)^3 2\omega} \]
Finally we have a real classical solution of the K.-G. equation:

\[ \varphi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega} \left[ a(k)e^{ikx} + a^*(k)e^{-ikx} \right] \]

where again: \( kx = k \cdot x - \omega t \), \( \omega = +(k^2 + m^2)^{1/2} \), \( \frac{d^3k}{(2\pi)^3 2\omega} \)

For later use we can express \( a(k) \) in terms of \( \varphi(x) \):

\[ \int d^3x e^{-ikx} \varphi(x) = \frac{1}{2\omega} a(k) + \frac{i}{2\omega} e^{2i\omega t} a^*(-k) , \]

\[ \int d^3x e^{-ikx} \partial_0 \varphi(x) = -\frac{i}{2} a(k) + \frac{i}{2} e^{2i\omega t} a^*(-k) . \]

\[ a(k) = \int d^3x e^{-ikx} \left[ i\partial_0 \varphi(x) + \omega \varphi(x) \right] = i \int d^3x e^{-ikx} \partial_0 \varphi(x) , \]

where \( \overrightarrow{\partial}_\mu g \equiv f(\partial_\mu g) - (\partial_\mu f)g \) and we will call \( \partial_0 \varphi = \partial \varphi / \partial t = \varphi \).

Note, \( a(k) \) is time independent.

**Constructing the hamiltonian:**

Recall, in classical mechanics, starting with lagrangian \( L(q_i, \dot{q}_i) \) as a function of coordinates \( q_i \) and their time derivatives \( \dot{q}_i \) we define conjugate momenta \( p_i = \partial L / \partial \dot{q}_i \) and the hamiltonian is then given as:

\[ H = \sum_i p_i \dot{q}_i - L \]

In field theory:

\[ q_i \quad \rightarrow \quad \varphi(x, t) \]

\[ p_i = \partial L / \partial \dot{q}_i \quad \rightarrow \quad \Pi(x) = \frac{\partial L}{\partial \varphi(x)} \]

\[ H = \sum_i p_i \dot{q}_i - L \quad \rightarrow \quad \mathcal{H} = \Pi \dot{\varphi} - \mathcal{L} \quad \text{hamiltonian density} \]

and the hamiltonian is given as: \( H = \int d^3 x \mathcal{H} \)
In our case:

\[
\mathcal{L} = -\frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} m^2 \varphi^2 + \Omega_0 \\
\Pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(x)} \\
\mathcal{H} = \frac{1}{2} \Pi^2 + \frac{1}{2} (\nabla \varphi)^2 + \frac{1}{2} m^2 \varphi^2 - \Omega_0
\]

Inserting \( \varphi(x) = \int d\vec{k} \left[ a(\vec{k}) e^{ikx} + a^*(\vec{k}) e^{-ikx} \right] \) we get:

\[
H = -\Omega_0 V + \frac{1}{2} \int d\vec{k} \int d\vec{k}' \int d^3x \left[ \\
\left( -i\omega a(\vec{k}) e^{ikx} + i\omega a^*(\vec{k}) e^{-ikx} \right) \left( -i\omega' a(\vec{k}') e^{ik'x} + i\omega' a^*(\vec{k}') e^{-ik'x} \right) \\
+ \left( +i\vec{k} a(\vec{k}) e^{ikx} - i\vec{k} a^*(\vec{k}) e^{-ikx} \right) \cdot \left( +i\vec{k}' a(\vec{k}') e^{ik'x} - i\vec{k}' a^*(\vec{k}') e^{-ik'x} \right) \\
+ m^2 \left( a(\vec{k}) e^{ikx} + a^*(\vec{k}) e^{-ikx} \right) \left( a(\vec{k}') e^{ik'x} + a^*(\vec{k}') e^{-ik'x} \right) \right]
\]

\[
H = -\Omega_0 V + \frac{1}{2} \int d\vec{k} d\vec{k}' d^3x \left[ \\
\left( -i\omega a(\vec{k}) e^{ikx} + i\omega a^*(\vec{k}) e^{-ikx} \right) \left( -i\omega a(\vec{k}') e^{ik'x} + i\omega a^*(\vec{k}') e^{-ik'x} \right) \\
+ \left( +i\vec{k} a(\vec{k}) e^{ikx} - i\vec{k} a^*(\vec{k}) e^{-ikx} \right) \cdot \left( +i\vec{k}' a(\vec{k}') e^{ik'x} - i\vec{k}' a^*(\vec{k}') e^{-ik'x} \right) \\
+ m^2 \left( a(\vec{k}) e^{ikx} + a^*(\vec{k}) e^{-ikx} \right) \left( a(\vec{k}') e^{ik'x} + a^*(\vec{k}') e^{-ik'x} \right) \right]
\]

\[
= -\Omega_0 V + \frac{1}{2} \frac{1}{(2\pi)^3} \int d\vec{k} \int \frac{1}{2\omega} \left[ \\
\delta^3(\vec{k} - \vec{k}') (+\omega\omega' + \vec{k} \cdot \vec{k}' + m^2) \\
\times \left( a^*(\vec{k}) a(\vec{k}') e^{i(\omega-\omega')t} + a(\vec{k}) a^*(\vec{k}') e^{-i(\omega-\omega')t} \right) \\
+ \delta^3(\vec{k} + \vec{k}') (-\omega\omega' - \vec{k} \cdot \vec{k}' + m^2) \\
\times \left( a(\vec{k}) a(\vec{k}') e^{-i(\omega+\omega')t} + a^*(\vec{k}) a^*(\vec{k}') e^{i(\omega+\omega')t} \right) \right]
\]

\[
\omega = (k^2 + m^2)^{1/2}
\]

\[
= -\Omega_0 V + \frac{1}{2} \frac{1}{(2\pi)^3} \int d\vec{k} \omega \left[ a^*(\vec{k}) a(\vec{k}) + a(\vec{k}) a^*(\vec{k}) \right],
\]
From classical to quantum (canonical quantization):
“coordinates” and “momenta” are promoted to operators satisfying canonical commutation relations:

\[
[\varphi(x, t), \varphi(x', t)] = 0,
\]
\[
[\Pi(x, t), \Pi(x', t)] = 0,
\]
\[
[\varphi(x, t), \Pi(x', t)] = i\delta^3(x - x')
\]
operators are taken at equal times in the Heisenberg picture

\[
a(k) = \int d^3x \ e^{-ikx} \left[ i\partial_0\varphi(x) + \omega\varphi(x) \right] \\
\Pi(x) = \dot{\varphi}(x)
\]

\[
[a(k), a(k')] = 0,
\]
\[
[a^\dagger(k), a^\dagger(k')] = 0,
\]
\[
[a(k), a^\dagger(k')] = (2\pi)^32\omega\delta^3(k - k')
\]

We have derived the classical hamiltonian:

\[
H = -\Omega_0V + \frac{1}{2} \int \tilde{d}k \ \omega \left( a^\dagger(k)a(k) + a(k)a^\dagger(k) \right)
\]

We kept ordering of a's unchanged, so that we can easily generalize it to quantum theory where classical functions will become operators that may not commute.

The hamiltonian of the quantum theory:

\[
a^\dagger(k) \rightarrow a^\dagger(k) \quad \quad [a(k), a^\dagger(k')] = (2\pi)^32\omega\delta^3(k - k')
\]

\[
H = \int \tilde{d}k \ \omega \ a^\dagger(k)a(k) + (E_0 - \Omega_0)V \\
(2\pi)^3\delta^3(0) = V
\]

see the formula for delta function

\[
E_0 = \frac{1}{2}(2\pi)^{-3} \int d^3k \ \omega \quad \text{is the total zero point energy per unit volume}
\]

we are free to choose: \( \Omega_0 = E_0 \)

the ground state has zero energy eigenvalue.
Summary:

\[ H = \int \frac{d^3k}{(2\pi)^3} \omega \ a^\dagger(k) a(k) \]

\[ \{a(k), a(k')\} = 0, \]
\[ \{a^\dagger(k), a^\dagger(k')\} = 0, \]
\[ \{a(k), a^\dagger(k')\} = (2\pi)^3 2\omega \delta^3(k - k') \]

is equivalent to:

\[ H = \int d^3p \ (p^2 c^2 + m^2 c^4)^{1/2} \ a^\dagger(p) a(p) \]

\[ \{a(p), a(p')\}_\mp = 0, \]
\[ \{a^\dagger(p), a^\dagger(p')\}_\mp = 0, \]
\[ \{a(p), a^\dagger(p')\}_\mp = \delta^3(p - p') \]

for: \[ a(k) = [(2\pi)^3 2\omega]^{1/2} \tilde{a}(k) \]

We have rederived the Hamiltonian of free relativistic bosons by quantization of a scalar field whose equation of motion is the Klein-Gordon equation (starting with manifestly Lorentz invariant lagrangian).

does not work for fermions, anticommutators lead to trivial Hamiltonian!