Path integral for interacting field

Let’s consider an interacting “phi-cubed” QFT:

\[ \mathcal{L} = -\frac{1}{2}Z_\phi \partial^\mu \phi \partial_\mu \phi - \frac{1}{2}Z_m m^2 \phi^2 + \frac{1}{6}Z_g g \phi^3 + Y \phi \]

with fields satisfying:

\[ \langle 0 | \phi(x) | 0 \rangle = 0 \quad \langle k | \phi(x) | 0 \rangle = e^{-ikx} \]

we want to evaluate the path integral for this theory:

\[ Z(J) \equiv \langle 0 | 0 \rangle_J = \int D\phi \; e^{i \int d^4x [\mathcal{L}_0 + \mathcal{L}_1 + J\phi]} \]

\[ \mathcal{L} = -\frac{1}{2}Z_\phi \partial^\mu \phi \partial_\mu \phi - \frac{1}{2}Z_m m^2 \phi^2 + \frac{1}{6}Z_g g \phi^3 + Y \phi \]

it can be also written as:

\[ Z(J) \equiv \langle 0 | 0 \rangle_J = \int D\phi \; e^{i \int d^4x [\mathcal{L}_0 + \mathcal{L}_1 + J\phi]} \]

\[ \exp \left[ \frac{i}{6} \int d^4x \mathcal{L}_1 \left( \frac{1}{\delta J(x)} \right) \int D\phi \; e^{i \int d^4x [\mathcal{L}_0 + J\phi]} \right] \]

\[ \mathcal{L}_0 = -\frac{1}{2}Z_\phi \partial^\mu \phi \partial_\mu \phi - \frac{1}{2}Z_m m^2 \phi^2 + \frac{1}{6}Z_g g \phi^3 \]

\[ Z_0(J) = \exp \left[ \frac{i}{6} \int d^4x \mathcal{L}_1 \left( \frac{1}{\delta J(x)} \right) \right] Z_0(0) \]

\[ Z(J) = e^{i \int d^4x \mathcal{L}_1 \left( \frac{1}{\delta J(x)} \right) \int D\phi \; e^{i \int d^4x [\mathcal{L}_0 + J\phi]} \]

\[ \propto e^{i \int d^4x \mathcal{L}_1 \left( \frac{1}{\delta J(x)} \right) Z_0(J)} \]

\[ \text{assumes } \mathcal{L}_0 = -\frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 \]

thus in the case of:

\[ \mathcal{L} = -\frac{1}{2}Z_\phi \partial^\mu \phi \partial_\mu \phi - \frac{1}{2}Z_m m^2 \phi^2 + \frac{1}{6}Z_g g \phi^3 + Y \phi \]

the perturbing lagrangian is:

\[ \mathcal{L}_1 = \frac{1}{4}Z_\phi \phi \partial^\mu \phi \partial_\mu \phi \]

\[ \mathcal{L}_{ct} = -\frac{1}{2}(Z_\phi - 1) \partial^\mu \phi \partial_\mu \phi - \frac{1}{2}(Z_m - 1) m^2 \phi^2 + Y \phi \]

counterterm lagrangian

in the limit \( g \to 0 \) we expect \( Y \to 0 \) and \( Z_t \to 1 \)

we will find \( Y = O(g) \) and \( Z_t = 1 + O(g^2) \)

Let’s look at \( Z(J) \) (ignoring counterterms for now).

Define:

\[ Z_1(J) \propto \exp \left[ \frac{i}{6} \int d^4x \mathcal{L}_1 \left( \frac{1}{\delta J(x)} \right) \right] Z_0(J) \]

exponentials defined by series expansion:

\[ Z_1(J) \propto \sum_{V=0}^{\infty} \frac{1}{V!} \left[ \frac{iZg}{6} \int d^4x \left( \frac{1}{\delta J(x)} \right) \right]^V \]

\[ \times \sum_{P=0}^{\infty} \frac{1}{P!} \left[ \frac{i}{2} \int d^4y d^4z J(y) \Delta(y-z) J(z) \right]^P \]

let’s look at a term with particular values of \( P \) (propagators) and \( V \) (vertices):

number of surviving sources, (after taking all derivatives) \( E \) (for external) is

\[ E = 2P - 3V \]

\( 3V \) derivatives can act on \( 2P \) sources in \( (2P)! / (2P-3V)! \) different ways

e.g. for \( V = 2, P = 3 \) there is 6! different terms
\( V = 2, E = 0 \ ( P = 3 ) \): 

\[
\frac{1}{2} \left[ \frac{iZ_g g}{6} \int d^4x \left( \frac{1}{i \delta J(x)} \right)^3 \right] \quad \frac{1}{2} \left[ \frac{iZ_g g}{6} \int d^4x \left( \frac{1}{i \delta J(x)} \right)^3 \right]
\] 

\[
\frac{1}{3!} \left[ \frac{i}{2} \int d^4y d^4z J(y) \Delta(y-z) J(z) \right] \frac{1}{2} \left[ \int d^4y d^4z J(y) \Delta(y-z) J(z) \right] \frac{1}{2} \left[ \int d^4y d^4z J(y) \Delta(y-z) J(z) \right]
\]

\[
= \frac{1}{2!} \left[ \frac{iZ_g g}{6} \int d^4x \left( \frac{1}{i \delta J(x)} \right)^3 \right] \quad \frac{1}{2} \left[ \frac{iZ_g g}{6} \int d^4x \left( \frac{1}{i \delta J(x)} \right)^3 \right]
\]

\[
\frac{1}{3!} \left[ \frac{i}{2} \int d^4y d^4z J(y) \Delta(y-z) J(z) \right] \frac{1}{2} \left[ \int d^4y d^4z J(y) \Delta(y-z) J(z) \right] \frac{1}{2} \left[ \int d^4y d^4z J(y) \Delta(y-z) J(z) \right]
\]

\[
= \frac{1}{2!} \left[ \frac{iZ_g g}{6} \int d^4x \left( \frac{1}{i \delta J(x)} \right)^3 \right] \quad \frac{1}{2} \left[ \frac{iZ_g g}{6} \int d^4x \left( \frac{1}{i \delta J(x)} \right)^3 \right]
\]

\[
\frac{1}{3!} \left[ \frac{i}{2} \int d^4y d^4z J(y) \Delta(y-z) J(z) \right] \frac{1}{2} \left[ \int d^4y d^4z J(y) \Delta(y-z) J(z) \right] \frac{1}{2} \left[ \int d^4y d^4z J(y) \Delta(y-z) J(z) \right]
\]

Feynman diagrams:
- a line segment stands for a propagator \( \frac{1}{i} \Delta(x-y) \)
- vertex joining three line segments stands for \( iZ_g g \int d^4x \)
- a filled circle at one end of a line segment stands for a source \( i \int dx J(x) \)

\[ \text{e.g. for } V = 1, E = 1 \]

What about those symmetry factors?

Symmetry factors are related to symmetries of Feynman diagrams...

\[ \text{Symmetry factors:} \]
- we can rearrange three derivatives without changing diagram
- we can rearrange three vertices
- we can rearrange two sources
- we can rearrange propagators

this in general results in overcounting of the number of terms that give the same result; this happens when some rearrangement of derivatives gives the same match up to sources as some rearrangement of sources; this is always connected to some symmetry property of the diagram; factor by which we overcounted is the symmetry factor
propagators can be rearranged in $3!$ ways, and all these rearrangements can be duplicated by exchanging the derivatives at the vertices.

Figure 9.1: All connected diagrams with $E = 0$ and $V = 2$.

the endpoints of each propagator can be swapped and the effect is duplicated by swapping the two vertices.

Figure 9.2: All connected diagrams with $E = 0$ and $V = 4$.

Figure 9.4: All connected diagrams with $E = 1$ and $V = 3$.

Figure 9.3: All connected diagrams with $E = 1$ and $V = 1$.

Figure 9.5: All connected diagrams with $E = 2$ and $V = 0$.

Figure 9.6: All connected diagrams with $E = 2$ and $V = 2$. 

$S = 2^3$

$S = 2 \times 3!$
Figure 9.7: All connected diagrams with $E = 2$ and $V = 4$.

Figure 9.8: All connected diagrams with $E = 3$ and $V = 1$.

Figure 9.9: All connected diagrams with $E = 3$ and $V = 3$.

Figure 9.10: All connected diagrams with $E = 4$ and $V = 2$.

Figure 9.11: All connected diagrams with $E = 4$ and $V = 4$. 
All these diagrams are connected, but $Z(J)$ contains also diagrams that are products of several connected diagrams:

e.g. for $V = 4, E = 0$ ($P = 6$) in addition to connected diagrams we also have:

and also:

and also:

Now $Z_1(J)$ is given by summing all diagrams $D$:

$$Z_1(J) \propto \sum_{\{n_i\}} D \propto \sum_{\{n_i\}} \prod_i \frac{1}{n_i!} \left( C_i \right)^{n_i}$$

$$\propto \prod_i \sum_{n_i = 0}^{\infty} \frac{1}{n_i!} \left( C_i \right)^{n_i}$$

$$\propto \prod_i \exp \left( C_i \right)$$

$$\propto \exp \left( \sum_i C_i \right).$$

thus we have found that $Z_2(J)$ is given by the exponential of the sum of connected diagrams.

imposing the normalization $Z_1(0) = 1$ means we can omit vacuum diagrams (those with no sources), thus we have:

$$Z_1(J) = \exp \left[ i W_1(J) \right], \quad i W_1(J) = \sum_{I \neq 0} C_I$$

vacuum diagrams are omitted from the sum

If there were no counterterms we would be done: $Z(J) = Z_1(J)$

in that case, the vacuum expectation value of the field is:

$$\langle 0 | \varphi(x) | 0 \rangle = \left. \frac{1}{i} \frac{\delta}{\delta J(x)} Z_1(J) \right|_{J=0}$$

$$= \left. \frac{\delta}{\delta J(x)} W_1(J) \right|_{J=0}.$$

only diagrams with one source contribute:

and we find:

$$\langle 0 | \varphi(x) | 0 \rangle = \frac{1}{2} i g \int dy \Delta(x-y) \Delta(y-x) + O(g^3)$$

we used $Z_g = 1$ since we know $Z_g = 1 + O(g^2)$

which is not zero, as required for the LSZ; so we need counterterm
Including $Y \varphi$ term in the interaction lagrangian results in a new type of vertex on which a line segment ends

\[ \text{e.g.} \quad S = 1 \quad S = 2 \quad S = 2 \quad S = 2 \]

corresponding Feynman rule is: \[ iY \int d^4y \]

at the lowest order of $g$ only \[ S = 1 \quad \text{contributes:} \]

\[ \langle 0 | \varphi(x) | 0 \rangle = \left( iY + \frac{1}{2}(ig)^{\gamma_5} \right) \int d^4y \frac{1}{4} \Delta(x-y) + O(g^3) \]

in order to satisfy \[ \langle 0 | \varphi(x) | 0 \rangle = 0 \] we have to choose:

\[ Y = \frac{1}{2} ig \Delta(0) + O(g^3) \]

\[ \Delta(0) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2 - i\epsilon} \]

Note, $\Delta(0)$ must be purely imaginary so that $Y$ is real; and, in addition, the integral over $k$ is ultraviolet divergent.

to make sense out of it, we introduce an ultraviolet cutoff \[ \Lambda \gg m \]

and in order to keep Lorentz-transformation properties of the propagator we make the replacement:

\[ \Delta(x-y) \rightarrow \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 + m^2 - i\epsilon} \left( \frac{\Lambda^2}{k^2 + \Lambda^2 - i\epsilon} \right)^2 \]

the integral is now convergent:

\[ \Delta(0) = \frac{i}{16\pi^2} \Lambda^2 \]

we will do this type of calculations later...

and indeed, $\Delta(0)$ is purely imaginary.

after choosing $Y$ so that $\langle 0 | \varphi(x) | 0 \rangle = 0$ we can take the limit $\Lambda \rightarrow \infty$ $Y$ becomes infinite

... we repeat the procedure at every order in $g$

e.g. at $O(g^3)$ we have to sum up:

\[ \text{e.g. at } O(g^3) \text{ we have to sum up:} \]

\[ \text{and add to } Y \text{ whatever } O(g^3) \text{ term is needed to maintain } \langle 0 | \varphi(x) | 0 \rangle = 0 \]

this way we can determine the value of $Y$ order by order in powers of $g$.

Adjusting $Y$ so that $\langle 0 | \varphi(x) | 0 \rangle = 0$ means that the sum of all connected diagrams with a single source is zero!

In addition, the same infinite set of diagrams with source replaced by ANY subdiagram is zero as well.

Rule: ignore any diagram that, when a single line is cut, fall into two parts, one of which has no sources. = \text{tadpoles}

all that is left with up to 4 sources and 4 vertices is:

\[ \text{Figure 9.13: All connected diagrams without tadpoles with } S \leq 4 \text{ and } V \leq 4. \]
we have calculated \( Z(J) \) in \( \varphi^3 \) theory and expressed it as

\[
Z(J) = \exp[iW(J)]
\]

where \( W \) is the sum of all connected diagrams with no tadpoles and at least two sources!

finally, let's take a look at the other two counterterms:

\[
\mathcal{L}_1 = \frac{1}{6} Z_\varphi \varphi^3 + \mathcal{L}_{ct},
\]

\[
\mathcal{L}_{ct} = -\frac{1}{2} (Z_m - 1) \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} (Z_m - 1) m^2 \varphi^2 + Y \varphi
\]

we get

\[
A = Z_\varphi - 1, \quad B = Z_m - 1
\]

\[
Z(J) = \exp\left[-\frac{i}{2} \int d^4x \left( \frac{1}{i \delta J(x)} \right) \left(-A \partial^2 + B m^2\right) \left( \frac{1}{i \delta J(x)} \right) \right] Z_2(J)
\]

we used integration by parts

it results in a new vertex at which two lines meet, the corresponding vertex factor or the Feynman rule is \((-i) \int d^4x \left(-A \partial^2 + B m^2\right)\)

for every diagram with a propagator there is additional one with this vertex

Summary:
we have calculated \( Z(J) \) in \( \varphi^3 \) theory and expressed it as

\[
Z(J) = \exp[iW(J)]
\]

where \( W \) is the sum of all connected diagrams with no tadpoles and at least two sources!