Scattering amplitudes and the Feynman rules

based on S-10

We have found Z(J) for the "phi-cubed" theory and now we can calculate vacuum expectation values of the time ordered products of any number of fields.

Let's define exact propagator:

$$\tfrac{1}{i}\mathbf{\Delta}(x_1-x_2) \equiv \langle 0|\mathrm{T}\varphi(x_1)\varphi(x_2)|0\rangle$$

short notation: $\delta_j \equiv \dfrac{1}{i}\dfrac{\delta}{\delta J(x_j)}$

$$\begin{aligned}\langle 0|\mathrm{T}\varphi(x_1)\varphi(x_2)|0\rangle &= \delta_1\delta_2 Z(J)\Big|_{J=0}\\ &= \delta_1\delta_2 iW(J)\Big|_{J=0} - \delta_1 iW(J)\Big|_{J=0}\delta_2 iW(J)\Big|_{J=0}\\ &= \delta_1\delta_2 iW(J)\Big|_{J=0}\,.\end{aligned}$$

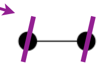
$Z(J) = \exp[iW(J)]$

$$\delta_j W(J)|_{J=0} = \langle 0|\varphi(x_j)|0\rangle = 0$$

W contains diagrams with at least two sources ●—● + ...

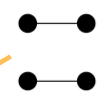
thus we find:

$$\tfrac{1}{i}\mathbf{\Delta}(x_1-x_2) = \tfrac{1}{i}\Delta(x_1-x_2) + O(g^2)$$



4-point function:

$$\begin{aligned}\langle 0|\mathrm{T}\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)|0\rangle &= \delta_1\delta_2\delta_3\delta_4 Z(J)\\ &= \Big[\,\delta_1\delta_2\delta_3\delta_4 iW\\ &\quad + (\delta_1\delta_2 iW)(\delta_3\delta_4 iW)\\ &\quad + (\delta_1\delta_3 iW)(\delta_2\delta_4 iW)\\ &\quad + (\delta_1\delta_4 iW)(\delta_2\delta_3 iW)\Big]_{J=0}\end{aligned}$$



we have dropped terms that contain $\langle 0|\varphi(x)|0\rangle = 0$

does not correspond to any interaction; when plugged to LSZ, no scattering happens

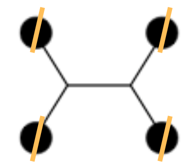
Let's define connected correlation functions:

$$\langle 0|\mathrm{T}\varphi(x_1)\ldots\varphi(x_E)|0\rangle_{\mathrm{C}} \equiv \delta_1\ldots\delta_E iW(J)\Big|_{J=0}$$

and plug these into LSZ formula.

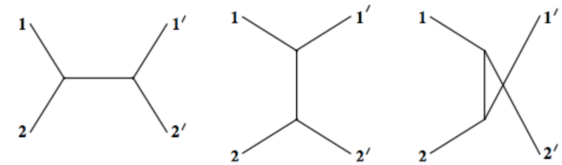
$$\langle 0|\mathrm{T}\varphi(x_1)\varphi(x_2)\varphi(x_1')\varphi(x_2')|0\rangle_{\mathrm{C}} = \delta_1\delta_2\delta_{1'}\delta_{2'} iW\Big|_{J=0}$$

at the lowest order in g only one diagram contributes:



S = 8

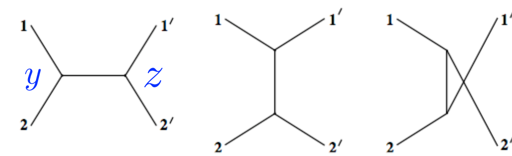
derivatives remove sources in 4! possible ways, and label external legs in 3 distinct ways:



each diagram occurs 8 times, which nicely cancels the symmetry factor.

General result for tree diagrams (no closed loops): each diagram with a distinct endpoint labeling has an overall symmetry factor 1.

Let's finish the calculation of $\langle 0|\mathrm{T}\varphi(x_1)\varphi(x_2)\varphi(x_1')\varphi(x_2')|0\rangle_{\mathrm{C}} = \delta_1\delta_2\delta_{1'}\delta_{2'} iW\Big|_{J=0}$



putting together factors for all pieces of Feynman diagrams we get:

$$\begin{aligned}&\langle 0|\mathrm{T}\varphi(x_1)\varphi(x_2)\varphi(x_1')\varphi(x_2')|0\rangle_{\mathrm{C}}\\ &\quad = (ig)^2\left(\tfrac{1}{i}\right)^5\int d^4y\,d^4z\,\Delta(y-z)\\ &\quad\times\Big[\,\Delta(x_1-y)\Delta(x_2-y)\Delta(x_1'-z)\Delta(x_2'-z)\\ &\quad + \Delta(x_1-y)\Delta(x_1'-y)\Delta(x_2-z)\Delta(x_2'-z)\\ &\quad + \Delta(x_1-y)\Delta(x_2'-y)\Delta(x_2-z)\Delta(x_1'-z)\,\Big]\\ &\quad + O(g^4)\,.\end{aligned}$$

For two incoming and two outgoing particles the LSZ formula is:

$$\langle f|i\rangle = i^4 \int d^4x_1\, d^4x_2\, d^4x_1'\, d^4x_2'\, e^{i(k_1x_1+k_2x_2-k_1'x_1'-k_2'x_2')}$$
$$\times(-\partial_1^2+m^2)(-\partial_2^2+m^2)(-\partial_{1'}^2+m^2)(-\partial_{2'}^2+m^2)$$
$$\times\langle 0|\mathrm{T}\varphi(x_1)\varphi(x_2)\varphi(x_1')\varphi(x_2')|0\rangle\ .$$

and we have just written $\langle 0|\mathrm{T}\varphi(x_1)\varphi(x_2)\varphi(x_1')\varphi(x_2')|0\rangle_\mathrm{C} = \delta_1\delta_2\delta_{1'}\delta_{2'}iW\big|_{J=0}$ in terms of propagators.

The LSZ formula highly simplifies due to:

$$(-\partial_i^2+m^2)\Delta(x_i-y) = \delta^4(x_i-y)$$

We find:

$$\langle f|i\rangle = (ig)^2\left(\tfrac{1}{i}\right)\int d^4y\, d^4z\, \Delta(y-z)\Big[\, e^{i(k_1y+k_2y-k_1'z-k_2'z)}$$
$$+ e^{i(k_1y+k_2z-k_1'y-k_2'z)}$$
$$+ e^{i(k_1y+k_2z-k_1'z-k_2'y)}\,\Big] + O(g^4)$$

$$\langle f|i\rangle = (ig)^2\left(\tfrac{1}{i}\right)\int d^4y\, d^4z\, \Delta(y-z)\Big[\, e^{i(k_1y+k_2y-k_1'z-k_2'z)}$$
$$+ e^{i(k_1y+k_2z-k_1'y-k_2'z)}$$
$$+ e^{i(k_1y+k_2z-k_1'z-k_2'y)}\,\Big] + O(g^4)$$

$$\Delta(y-z) = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik(y-z)}}{k^2+m^2-i\epsilon}$$

$$\langle f|i\rangle = ig^2\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2+m^2-i\epsilon}$$
$$\times\Big[\,(2\pi)^4\delta^4(k_1+k_2+k)\,(2\pi)^4\delta^4(k_1'+k_2'+k)$$
$$+(2\pi)^4\delta^4(k_1-k_1'+k)\,(2\pi)^4\delta^4(k_2'-k_2+k)$$
$$+(2\pi)^4\delta^4(k_1-k_2'+k)\,(2\pi)^4\delta^4(k_1'-k_2+k)\,\Big] + O(g^4)$$
$$= ig^2\,(2\pi)^4\delta^4(k_1+k_2-k_1'-k_2')$$
$$\times\left[\frac{1}{(k_1+k_2)^2+m^2}+\frac{1}{(k_1-k_1')^2+m^2}+\frac{1}{(k_1-k_2')^2+m^2}\right]$$
$$+O(g^4)\ .$$

$m^2 - i\epsilon$

four-momentum is conserved in scattering process

$$\langle f|i\rangle = ig^2\,(2\pi)^4\delta^4(k_1+k_2-k_1'-k_2')$$
$$\times\left[\frac{1}{(k_1+k_2)^2+m^2}+\frac{1}{(k_1-k_1')^2+m^2}+\frac{1}{(k_1-k_2')^2+m^2}\right]$$
$$+O(g^4)\ .$$

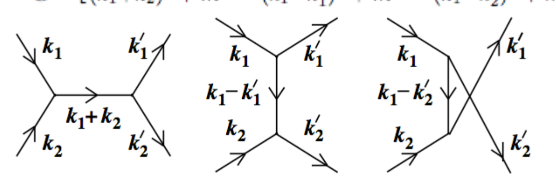
Let's define:

$$\langle f|i\rangle = (2\pi)^4\delta^4(k_\mathrm{in}-k_\mathrm{out})\,i\mathcal{T}$$

scattering matrix element

From this calculation we can deduce a set of rules for computing $i\mathcal{T}$.

$$i\mathcal{T} = ig^2\left[\frac{1}{(k_1+k_2)^2+m^2}+\frac{1}{(k_1-k_1')^2+m^2}+\frac{1}{(k_1-k_2')^2+m^2}\right]+O(g^4)$$



Feynman rules to calculate $i\mathcal{T}$:

- for each incoming and outgoing particle draw an external line and label it with four-momentum and an arrow specifying the momentum flow
- draw all topologically inequivalent diagrams
- for internal lines draw arrows arbitrarily but label them with momenta so that momentum is conserved in each vertex
- assign factors:

1	for each external line
$-i/(k^2+m^2-i\epsilon)$	for each internal line with momentum k
$iZ_g g$	for each vertex

- sum over all the diagrams and get $i\mathcal{T}$

Additional rules for diagrams with loops:

- a diagram with L loops will have L internal momenta that are not fixed; integrate over all these momenta with measure

$$d^4\ell_i/(2\pi)^4$$

- divide by a symmetry factor
- include diagrams with counterterm vertex that connects two propagators, each with the same momentum k; the value of the vertex is

$$-i(Ak^2 + Bm^2)$$

$$A = Z_\varphi - 1$$
$$B = Z_m - 1$$

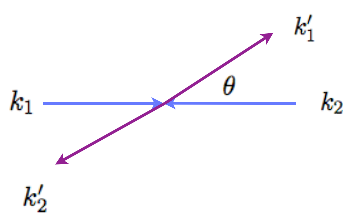
now we are going to use $i\mathcal{T}$ to calculate cross section...

Cross sections and decay rates

based on S-11

Particle physics experiments typically measure cross sections and decay rates.

Kinematics of a scattering process:



$$k_1 + k_2 = k_1' + k_2'$$
$$k_i^2 = -m_i^2$$

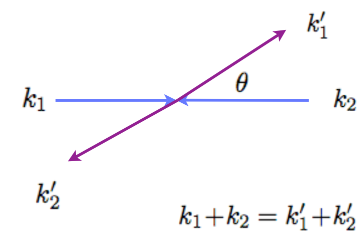
particles can have different mass

two convenient frames:

- center-of-mass, or CM frame: $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{0}$
- fixed target, or FT frame (lab frame): $\mathbf{k}_2 = \mathbf{0}$

CM frame (we choose k_1 to be in z-direction):

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{0}$$



$$k_1 + k_2 = k_1' + k_2'$$
$$k_i^2 = -m_i^2$$

there is only one free initial parameter $|\mathbf{k}_1|$.

However it is convenient to define:

$$s \equiv -(k_1 + k_2)^2$$

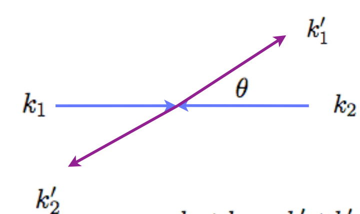
which is Lorentz invariant; in the CM frame it is equal $(E_1 + E_2)^2$

center-of-mass energy squared

$$E_1 = (\mathbf{k}_1^2 + m_1^2)^{1/2}$$
$$E_2 = (\mathbf{k}_1^2 + m_2^2)^{1/2}$$

then we find:

$$|\mathbf{k}_1| = \frac{1}{2\sqrt{s}}\sqrt{s^2 - 2(m_1^2 + m_2^2)s + (m_1^2 - m_2^2)^2} \quad \text{(CM frame)}$$



$$k_1 + k_2 = k_1' + k_2'$$
$$k_i^2 = -m_i^2$$

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{0}$$

$$s \equiv -(k_1 + k_2)^2$$

$$\mathbf{k}_1' + \mathbf{k}_2' = \mathbf{0}$$

$$(E_1' + E_2')^2 = s$$

$$|\mathbf{k}_1'| = \frac{1}{2\sqrt{s}}\sqrt{s^2 - 2(m_{1'}^2 + m_{2'}^2)s + (m_{1'}^2 - m_{2'}^2)^2} \quad \text{(CM frame)}$$

finally instead of θ , it is convenient to define a Lorentz scalar:

$$t \equiv -(k_1 - k_1')^2$$

$$t = m_1^2 + m_{1'}^2 - 2E_1E_1' + 2|\mathbf{k}_1||\mathbf{k}_1'|\cos\theta$$

Mandelstam variables:

$$s \equiv -(k_1+k_2)^2 = -(k_1'+k_2')^2$$
$$t \equiv -(k_1-k_1')^2 = -(k_2-k_2')^2$$
$$u \equiv -(k_1-k_2')^2 = -(k_2-k_1')^2$$

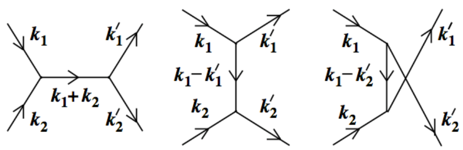
they satisfy:

$$s+t+u = m_1^2+m_2^2+m_{1'}^2+m_{2'}^2$$

the scattering matrix element in φ^3 can be simply written as:

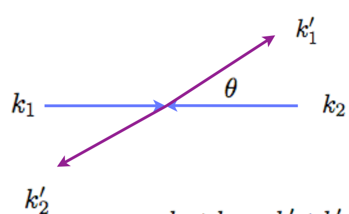
$$i\mathcal{T} = ig^2\left[\frac{1}{(k_1+k_2)^2+m^2}+\frac{1}{(k_1-k_1')^2+m^2}+\frac{1}{(k_1-k_2')^2+m^2}\right]$$

$$\mathcal{T} = g^2\left[\frac{1}{m^2-s}+\frac{1}{m^2-t}+\frac{1}{m^2-u}\right]+O(g^4)$$



Fixed target frame:

$$\mathbf{k}_2 = \mathbf{0}$$



$$k_1+k_2 = k_1'+k_2'$$
$$k_i^2 = -m_i^2$$

in this case from $s \equiv -(k_1+k_2)^2$ we have:

$$|\mathbf{k}_1| = \frac{1}{2m_2}\sqrt{s^2-2(m_1^2+m_2^2)s+(m_1^2-m_2^2)^2} \quad \text{(FT frame)}$$

comparing it with the result in the CM frame,

$$|\mathbf{k}_1| = \frac{1}{2\sqrt{s}}\sqrt{s^2-2(m_1^2+m_2^2)s+(m_1^2-m_2^2)^2} \quad \text{(CM frame)}$$

we find

$$m_2|\mathbf{k}_1|_{\rm FT} = \sqrt{s}\,|\mathbf{k}_1|_{\rm CM}$$

Formula for the differential scattering cross section:

we assume the experiment is taking place in a big box of volume V, and lasts for a large time T (we should be thinking about colliding wave packets but we will simplify the discussion somewhat, for more precise treatment see e.g. Peskin and Schroeder)

probability for 1,2 → 1',2'...,n' is:

$$\langle f|i\rangle = (2\pi)^4\delta^4(k_{\rm in}-k_{\rm out})i\mathcal{T}$$

$$P = \frac{|\langle f|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle}$$

$$|\langle f|i\rangle|^2 = [(2\pi)^4\delta^4(k_{\rm in}-k_{\rm out})]^2\,|\mathcal{T}|^2$$

norm of a single particle state is:

$$\langle k|k\rangle = (2\pi)^3 2k^0\delta^3(\mathbf{0}) = 2k^0 V$$

$$[(2\pi)^4\delta^4(k_{\rm in}-k_{\rm out})]^2 = (2\pi)^4\delta^4(k_{\rm in}-k_{\rm out})\times(2\pi)^4\delta^4(0)$$

thus we have:

$$\langle i|i\rangle = 4E_1E_2V^2$$

$$(2\pi)^4\delta^4(0) = \int d^4x\, e^{i0\cdot x} = VT$$

$$\langle f|f\rangle = \prod_{j=1}^{n'} 2k_j'^0 V$$

$$\dot{P} = \frac{(2\pi)^4\delta^4(k_{\rm in}-k_{\rm out})\,V\,|\mathcal{T}|^2}{4E_1E_2V^2\prod_{j=1}^{n'}2k_j'^0V}$$

probability per unit time

$$\dot{P} = \frac{(2\pi)^4\delta^4(k_{\rm in}-k_{\rm out})\,V\,|\mathcal{T}|^2}{4E_1E_2V^2\prod_{j=1}^{n'}2k_j'^0V}$$

this is the probability per unit time to scatter into a set of outgoing particles with precise momenta.

we should sum over each momenta $\mathbf{k}_j'$ in a small range; due to the box we have:

vector with integer entries

$$\mathbf{k}_j' = (2\pi/L)\mathbf{n}_j'$$

$$V = L^3$$

in the limit of large L we have:

$$\sum_{\mathbf{n}_j'} \to \frac{V}{(2\pi)^3}\int d^3k_j'$$

thus we should consider:

$$\widetilde{dk} \equiv \frac{d^3k}{(2\pi)^3 2k^0}$$

$$\dot{P} = \frac{(2\pi)^4\delta^4(k_{\rm in}-k_{\rm out})}{4E_1E_2V}\,|\mathcal{T}|^2\prod_{j=1}^{n'}\widetilde{dk}_j'$$

$$\dot{P} = \frac{(2\pi)^4 \delta^4(k_{\rm in} - k_{\rm out})}{4E_1 E_2 V} |\mathcal{T}|^2 \prod_{j=1}^{n'} \widetilde{dk}'_j$$

finally to get the cross section we should divide by the incident flux:

cross section x incident flux = Probability per unit time

↓

= the number of particles per unit volume that are striking the target particle times their speed (easy to evaluate in the FT frame):

we have one particle in V with speed $v = |\mathbf{k}_1|/E_1$ and so the incident flux is $|\mathbf{k}_1|/E_1 V$

thus in the CM frame (using $m_2|\mathbf{k}_1|_{\rm FT} = \sqrt{s}\,|\mathbf{k}_1|_{\rm CM}$ and $E_2 = m_2$) we find:

$$d\sigma = \frac{1}{4|\mathbf{k}_1|_{\rm CM}\sqrt{s}} |\mathcal{T}|^2 \, d{\rm LIPS}_{n'}(k_1 + k_2)$$

where we defined the n'-body Lorentz-invariant phase-space measure:

$$d{\rm LIPS}_{n'}(k) \equiv (2\pi)^4 \delta^4(k - \textstyle\sum_{j=1}^{n'} k'_i) \displaystyle\prod_{j=1}^{n'} \widetilde{dk}'_j$$

Two outgoing particles:

$$d{\rm LIPS}_2(k) = (2\pi)^4 \delta^4(k - k'_1 - k'_2) \, \widetilde{dk}'_1 \widetilde{dk}'_2$$

$$k = k_1 + k_2$$

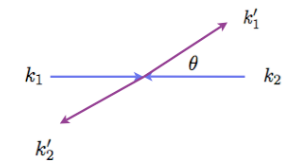
Lorentz invariant, we can compute it in any frame,
it is convenient to work in the CM frame:

$$\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{0} \qquad k^0 = E_1 + E_2 = \sqrt{s}$$

$$d{\rm LIPS}_2(k) = \frac{1}{4(2\pi)^2 E'_1 E'_2} \delta(E'_1 + E'_2 - \sqrt{s}\,) \, \delta^3(\mathbf{k}'_1 + \mathbf{k}'_2) \, d^3k'_1 d^3k'_2$$

$$d{\rm LIPS}_2(k) = \frac{1}{4(2\pi)^2 E'_1 E'_2} \delta(E'_1 + E'_2 - \sqrt{s}\,) \, d^3k'_1$$

$$d{\rm LIPS}_2(k) = \frac{1}{4(2\pi)^2 E'_1 E'_2} \delta(E'_1 + E'_2 - \sqrt{s}\,) \, d^3k'_1$$



$$E'_1 = \sqrt{\mathbf{k}'^2_1 + m^2_{1'}} \qquad E'_2 = \sqrt{\mathbf{k}'^2_1 + m^2_{2'}}$$

$$d^3k'_1 = |\mathbf{k}'_1|^2 \, d|\mathbf{k}'_1| \, d\Omega_{\rm CM} \qquad d\Omega_{\rm CM} = \sin\theta \, d\theta \, d\phi$$

differential solid angle

can be evaluated using

$$\textstyle\int dx \, \delta(f(x)) = \sum_i |f'(x_i)|^{-1}$$

in our case f(x)=0 for

$$|\mathbf{k}'_1| = \frac{1}{2\sqrt{s}} \sqrt{s^2 - 2(m^2_{1'} + m^2_{2'})s + (m^2_{1'} - m^2_{2'})^2}$$

$$\begin{aligned} \frac{\partial}{\partial|\mathbf{k}'_1|}\left(E'_1 + E'_2 - \sqrt{s}\right) &= \frac{|\mathbf{k}'_1|}{E'_1} + \frac{|\mathbf{k}'_1|}{E'_2} \\ &= |\mathbf{k}'_1|\left(\frac{E'_1 + E'_2}{E'_1 E'_2}\right) \\ &= \frac{|\mathbf{k}'_1|\sqrt{s}}{E'_1 E'_2} \, . \end{aligned}$$

thus we have:

$$d{\rm LIPS}_2(k) = \frac{|\mathbf{k}'_1|}{16\pi^2\sqrt{s}} \, d\Omega_{\rm CM}$$

$$d\sigma = \frac{1}{4|\mathbf{k}_1|_{\rm CM}\sqrt{s}} |\mathcal{T}|^2 \, d{\rm LIPS}_{n'}(k_1 + k_2) \qquad d{\rm LIPS}_2(k) = \frac{|\mathbf{k}'_1|}{16\pi^2\sqrt{s}} \, d\Omega_{\rm CM}$$

$$\frac{d\sigma}{d\Omega_{\rm CM}} = \frac{1}{64\pi^2 s} \frac{|\mathbf{k}'_1|}{|\mathbf{k}_1|} |\mathcal{T}|^2$$

$$d\Omega_{\rm CM} = \sin\theta \, d\theta \, d\phi$$

$$|\mathbf{k}_1| = \frac{1}{2\sqrt{s}} \sqrt{s^2 - 2(m^2_1 + m^2_2)s + (m^2_1 - m^2_2)^2}$$

$$|\mathbf{k}'_1| = \frac{1}{2\sqrt{s}} \sqrt{s^2 - 2(m^2_{1'} + m^2_{2'})s + (m^2_{1'} - m^2_{2'})^2}$$

or, in a frame independent form:

$$t = m^2_1 + m^2_{1'} - 2E_1 E'_1 + 2|\mathbf{k}_1||\mathbf{k}'_1|\cos\theta$$

$$\begin{aligned} dt &= 2\,|\mathbf{k}_1|\,|\mathbf{k}'_1| \, d\cos\theta \\ &= 2\,|\mathbf{k}_1|\,|\mathbf{k}'_1| \, \frac{d\Omega_{\rm CM}}{2\pi} \end{aligned}$$

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s |\mathbf{k}_1|^2} |\mathcal{T}|^2$$

in general, $|\mathbf{k}'_1|$ depends on both **s** and θ and so
the formula is more complicated than in the CM frame

Total cross section:

$$\sigma = \frac{1}{S}\int d\sigma$$

the number of identical outgoing particles of type i

$$S = \prod_i n_i'!$$

symmetry factor

dLIPS treats outgoing particles as an ordered list of momenta

For two outgoing particles we have:

$$\sigma = \frac{1}{S}\int d\Omega_{\rm CM}\frac{d\sigma}{d\Omega_{\rm CM}}$$

$$= \frac{2\pi}{S}\int_{-1}^{+1} d\cos\theta \, \frac{d\sigma}{d\Omega_{\rm CM}}$$

or, equivalently:

$$\sigma = \frac{1}{S}\int_{t_{\rm min}}^{t_{\rm max}} dt\,\frac{d\sigma}{dt}$$

correspond to $\cos\theta = -1$ and $+1$.

Let's get back to the scattering process in φ^3 theory we considered:

$$\mathcal{T} = g^2\left[\frac{1}{m^2-s}+\frac{1}{m^2-t}+\frac{1}{m^2-u}\right] + O(g^4)$$

In the CM frame:

all masses equal

$$E = \tfrac{1}{2}\sqrt{s}$$

$$|\mathbf{k}_1'| = |\mathbf{k}_1| = \tfrac{1}{2}(s-4m^2)^{1/2}$$

$$|\mathbf{k}_1| = \frac{1}{2\sqrt{s}}\sqrt{s^2-2(m_1^2+m_2^2)s+(m_1^2-m_2^2)^2}$$

$$|\mathbf{k}_1'| = \frac{1}{2\sqrt{s}}\sqrt{s^2-2(m_{1'}^2+m_{2'}^2)s+(m_{1'}^2-m_{2'}^2)^2}$$

$$t = -\tfrac{1}{2}(s-4m^2)(1-\cos\theta)$$

$$t = m_1^2+m_{1'}^2-2E_1E_1'+2|\mathbf{k}_1||\mathbf{k}_1'|\cos\theta$$

$$u = -\tfrac{1}{2}(s-4m^2)(1+\cos\theta)$$

we obtain $|\mathcal{T}|^2$ as a complicated function of s and θ .

In the nonrelativistic limit, $|\mathbf{k}_1| \ll m$ or $s-4m^2 \ll m^2$:

$$\mathcal{T} = \frac{5g^2}{3m^2}\left[1-\frac{8}{15}\left(\frac{s-4m^2}{m^2}\right)+\frac{5}{18}\left(1+\frac{27}{25}\cos^2\theta\right)\left(\frac{s-4m^2}{m^2}\right)^2+\dots\right]$$

differential cross section almost isotropic.

Let's get back to the scattering process in φ^3 theory we considered:

$$\mathcal{T} = g^2\left[\frac{1}{m^2-s}+\frac{1}{m^2-t}+\frac{1}{m^2-u}\right] + O(g^4)$$

In the CM frame:

all masses equal

$$E = \tfrac{1}{2}\sqrt{s}$$

$$|\mathbf{k}_1'| = |\mathbf{k}_1| = \tfrac{1}{2}(s-4m^2)^{1/2}$$

$$|\mathbf{k}_1| = \frac{1}{2\sqrt{s}}\sqrt{s^2-2(m_1^2+m_2^2)s+(m_1^2-m_2^2)^2}$$

$$|\mathbf{k}_1'| = \frac{1}{2\sqrt{s}}\sqrt{s^2-2(m_{1'}^2+m_{2'}^2)s+(m_{1'}^2-m_{2'}^2)^2}$$

$$t = -\tfrac{1}{2}(s-4m^2)(1-\cos\theta)$$

$$t = m_1^2+m_{1'}^2-2E_1E_1'+2|\mathbf{k}_1||\mathbf{k}_1'|\cos\theta$$

$$u = -\tfrac{1}{2}(s-4m^2)(1+\cos\theta)$$

we obtain $|\mathcal{T}|^2$ as a complicated function of s and θ .

In the extreme relativistic limit, $|\mathbf{k}_1| \gg m$ or $s \gg m^2$:

$$\mathcal{T} = \frac{g^2}{s\sin^2\theta}\left[3+\cos^2\theta-\left(\frac{(3+\cos^2\theta)^2}{\sin^2\theta}-16\right)\frac{m^2}{s}+\dots\right]$$

diff. cross section sharply peaked in the forward and backward directions.

$\theta = 0$ $\qquad$ $\theta = \pi$

Let's get back to the scattering process in φ^3 theory we considered:

$$\mathcal{T} = g^2\left[\frac{1}{m^2-s}+\frac{1}{m^2-t}+\frac{1}{m^2-u}\right] + O(g^4)$$

integrating over t (for fixed s) we can calculate the cross section:

$$t_{\rm min} = -(s-4m^2) \qquad t_{\rm max} = 0 \qquad t = -\tfrac{1}{2}(s-4m^2)(1-\cos\theta)$$

correspond to $\cos\theta = -1$ and $+1$.

$$\sigma = \frac{1}{S}\int_{t_{\rm min}}^{t_{\rm max}} dt\,\frac{d\sigma}{dt}$$

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s|\mathbf{k}_1|^2}|\mathcal{T}|^2$$

$$|\mathbf{k}_1'| = |\mathbf{k}_1| = \tfrac{1}{2}(s-4m^2)^{1/2}$$

$$u = 4m^2-s-t$$

$$S = 2$$

we get:

$$\sigma = \frac{g^4}{32\pi s(s-4m^2)}\left[\frac{2}{m^2}+\frac{s-4m^2}{(s-m^2)^2}-\frac{2}{s-3m^2} + \frac{4m^2}{(s-m^2)(s-2m^2)}\ln\left(\frac{s-3m^2}{m^2}\right)\right] + O(g^6)$$

$$\sigma = \frac{g^4}{32\pi s(s-4m^2)}\left[\frac{2}{m^2} + \frac{s-4m^2}{(s-m^2)^2} - \frac{2}{s-3m^2} + \frac{4m^2}{(s-m^2)(s-2m^2)}\ln\left(\frac{s-3m^2}{m^2}\right)\right] + O(g^6)$$

In the nonrelativistic limit, $|\mathbf{k}_1| \ll m$ or $s - 4m^2 \ll m^2$:

$$\sigma = \frac{25g^4}{1152\pi m^6}\left[1 - \frac{79}{60}\left(\frac{s-4m^2}{m^2}\right) + \dots\right] + O(g^6)$$

In the extreme relativistic limit, $|\mathbf{k}_1| \gg m$ or $s \gg m^2$:

$$\sigma = \frac{g^4}{16\pi m^2 s^2}\left[1 + \frac{7}{2}\frac{m^2}{s} + \dots\right] + O(g^6)$$

Formula for the differential decay rate:

we assume that the LSZ formula is valid for a single particle that can decay

following the derivation of $d\sigma$:

$$\dot{P} = \frac{(2\pi)^4\delta^4(k_{\rm in} - k_{\rm out})}{4E_1E_2V}|\mathcal{T}|^2 \prod_{j=1}^{n'} \widetilde{dk}'_j$$

with the only difference being:

$$P = \frac{|\langle f|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle}$$

$$\langle i|i\rangle = 2E_1V$$

$$\langle i|i\rangle = 4E_1E_2V^2$$

$$\langle f|f\rangle = \prod_{j=1}^{n'} 2k'^0_j V$$

identifying $d\Gamma$ with $\dot{P}$ gives:

$$d\Gamma = \frac{1}{2E_1}|\mathcal{T}|^2\, d\text{LIPS}_{n'}(k_1)$$

$s = -k_1^2 = m_1^2$

In the CM frame $E_1 = m_1$; in other frames, the relative factor E_1/m_1 accounts for relativistic time dilation of the decay rate.

Finally, a total decay rate:

$$\Gamma = \frac{1}{S}\int d\Gamma$$