Continuous symmetries and conserved currents

Consider a set of scalar fields $\varphi_a(x)$, and a lagrangian density

$$\mathcal{L}(x) = \mathcal{L}(\varphi_a(x), \partial_\mu \varphi_a(x))$$

let’s make an infinitesimal change: $\varphi_a(x) \rightarrow \varphi_a(x) + \delta \varphi_a(x)$

$$\mathcal{L}(x) \rightarrow \mathcal{L}(x) + \delta \mathcal{L}(x)$$

variation of the action:

$$\frac{\delta S}{\delta \varphi_a(x)} = \int d^4y \frac{\delta \mathcal{L}(y)}{\delta \varphi_a(x)}$$

$$= \int d^4y \left[ \frac{\partial \mathcal{L}(y)}{\partial \varphi_b(y)} \frac{\delta \varphi_b(y)}{\delta \varphi_a(x)} + \frac{\partial \mathcal{L}(y)}{\partial (\partial_\mu \varphi_b(y))} \frac{\delta (\partial_\mu \varphi_b(y))}{\delta \varphi_a(x)} \right]$$

$$= \int d^4y \left[ \frac{\partial \mathcal{L}(y)}{\partial \varphi_b(y)} \delta_{ba} \delta^4(y-x) + \frac{\partial \mathcal{L}(y)}{\partial (\partial_\mu \varphi_b(y))} \delta_{ba} \partial_\mu \delta^4(y-x) \right]$$

$$= \frac{\partial \mathcal{L}(x)}{\partial \varphi_a(x)} - \partial_\mu \frac{\partial \mathcal{L}(x)}{\partial (\partial_\mu \varphi_a(x))} \cdot \frac{\delta \mathcal{L}(x)}{\delta \varphi_a(x)} \rightarrow \partial_\mu \frac{\partial \mathcal{L}(x)}{\partial (\partial_\mu \varphi_a(x))} + \frac{\delta S}{\delta \varphi_a(x)}$$

setting $\frac{\delta S}{\delta \varphi_a(x)} = 0$ we would get equations of motion
If a set of infinitesimal transformations leaves the lagrangian unchanged, invariant, the Noether current is conserved. This is called the Noether current; now we have:

\[ \delta L(x) = \partial_\mu \left( \frac{\partial L(x)}{\partial (\partial_\mu \phi_a(x))} \delta \phi_a(x) \right) + \frac{\delta S}{\delta \phi_a(x)} \delta \phi_a(x) \]

\[ j^\mu(x) \equiv \frac{\partial L(x)}{\partial (\partial_\mu \phi_a(x))} \delta \phi_a(x) \]

This is called the Noether current; now we have:

\[ \partial_\mu j^\mu(x) = \delta L(x) - \frac{\delta S}{\delta \phi_a(x)} \delta \phi_a(x) \]

If a set of infinitesimal transformations leaves the lagrangian unchanged, invariant, \( \delta L = 0 \), the Noether current is conserved!

\[ \partial_\mu j^\mu = 0 \]

\[ \frac{\partial}{\partial t} j^0(x) + \nabla \cdot j(x) = 0 \]

charge density

current density
we set

\[ \hbar = c = 1 \]

any quantity \( A \) has units of mass to some power that we call \( [A] \), e.g:

\[
[m] = +1 \\
[x^\mu] = -1 \\
[\partial^\mu] = +1 \\
[d^dx] = -d
\]

it allows us to convert a time \( T \) to a length \( L \):

\[ T = c^{-1}L \]

a length to an inverse mass:

\[ L = \hbar c^{-1}M^{-1} \]

in \( d \) spacetime dimensions

the action appears in the exponential and so

\[ [S] = 0 \]

and for the lagrangian density we have:

\[ [\mathcal{L}] = d \]

\[
Z(J) = \int \mathcal{D}\varphi \exp \left[ i \int d^dx (\mathcal{L} + J\varphi) \right] \\
S = \int d^dx \mathcal{L}
\]
from the kinetic term:

$$\mathcal{L} = -\frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} m^2 \varphi^2 + \frac{1}{6} g \varphi^3$$

$$[\varphi] = \frac{1}{2} (d - 2)$$

in 4 dimensions:

$$[\varphi] = 1$$

functional derivative:

$$\frac{\delta}{\delta f(t_1)} f(t_2) = \delta(t_1 - t_2)$$

- dim = f
- dim = l
- dim = -f + l
similarly, in 4d:

\[
\frac{\delta \varphi_b(y)}{\delta \varphi_a(x)} = \delta_{ba} \delta^4(y-x)
\]

Noether current:

\[
\partial_\mu j^\mu(x) = \delta \mathcal{L}(x) - \frac{\delta S}{\delta \varphi_a(x)} \frac{\delta \varphi_a(x)}{\delta \varphi_a(x)}
\]

and so on ...
Consider a theory of a complex scalar field:

\[ \mathcal{L} = -\partial^\mu \varphi^\dagger \partial_\mu \varphi - m^2 \varphi^\dagger \varphi - \frac{1}{4} \lambda (\varphi^\dagger \varphi)^2 \]

clearly \( \mathcal{L} \) is left invariant by:

\[ \varphi(x) \rightarrow e^{-i\alpha} \varphi(x) \]

(U(1) transformation (transformation by a unitary 1x1 matrix))

in terms of two real scalar fields we get:

\[ \varphi = (\varphi_1 + i\varphi_2)/\sqrt{2} \]

\[ \mathcal{L} = -\frac{1}{2} \partial^\mu \varphi_1 \partial_\mu \varphi_1 - \frac{1}{2} \partial^\mu \varphi_2 \partial_\mu \varphi_2 - \frac{1}{2} m^2 (\varphi_1^2 + \varphi_2^2) - \frac{1}{16} \lambda (\varphi_1^2 + \varphi_2^2)^2 \]

and the U(1) transformation above is equivalent to:

\[
\begin{pmatrix}
\varphi_1(x) \\
\varphi_2(x)
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\cos \alpha & \sin \alpha \\
-\sin \alpha & \cos \alpha
\end{pmatrix}
\begin{pmatrix}
\varphi_1(x) \\
\varphi_2(x)
\end{pmatrix}
\]

(SO(2) transformation (transformation by an orthogonal 2x2 matrix with determinant = +1))
infinitesimal form of \( \varphi(x) \rightarrow e^{-i\alpha\varphi(x)} \) is:

\[
\begin{align*}
\varphi(x) & \rightarrow \varphi(x) - i\alpha\varphi(x) , \\
\varphi^\dagger(x) & \rightarrow \varphi^\dagger(x) + i\alpha\varphi^\dagger(x) ,
\end{align*}
\]

and the current is:

\[
\begin{align*}
\alpha j^\mu &= \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \delta \varphi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi^\dagger)} \delta \varphi^\dagger \\
&= (-\partial^\mu \varphi^\dagger)(-i\alpha\varphi) + (-\partial^\mu \varphi)(+i\alpha\varphi^\dagger) \\
&= \alpha \text{Im}(\varphi^\dagger \partial^\mu \varphi) ,
\end{align*}
\]

\[ A\partial^\mu B \equiv A\partial^\mu B - (\partial^\mu A)B \]
repeating the same for the SO(2) transformation:

\[
\delta \varphi_1 = +\alpha \varphi_2 \\
\delta \varphi_2 = -\alpha \varphi_1
\]

the Noether current is:

\[
\mathcal{L} = -\frac{1}{2} \partial^\mu \varphi_1 \partial_\mu \varphi_1 - \frac{1}{2} \partial^\mu \varphi_2 \partial_\mu \varphi_2 - \frac{1}{2} m^2 (\varphi_1^2 + \varphi_2^2) - \frac{1}{16} \lambda (\varphi_1^2 + \varphi_2^2)^2 \\
\jmath^\mu(x) \equiv \frac{\partial \mathcal{L}(x)}{\partial (\partial_\mu \varphi_a(x))} \delta \varphi_a(x)
\]

\[
\alpha \jmath^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi_1)} \delta \varphi_1 + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi_2)} \delta \varphi_2 \\
= (-\partial^\mu \varphi_1)(+\alpha \varphi_2) + (-\partial^\mu \varphi_2)(-\alpha \varphi_1) \\
= \alpha (\varphi_1 \overset{\rightarrow}{\partial^\mu} \varphi_2),
\]

which is equivalent to

\[
\jmath^\mu = \text{Im}(\varphi^\dagger \overset{\rightarrow}{\partial^\mu} \varphi) \\
\varphi = (\varphi_1 + i\varphi_2)/\sqrt{2}
\]
Let's define the Noether charge:

\[ Q \equiv \int d^3x \ j^0(x) = \int d^3x \ \text{Im}(\varphi^\dagger \tilde{\partial}^0 \varphi) \]

integrating \( \partial_t j^0(x) + \nabla \cdot j(x) = 0 \) over \( d^3x \), using Gauss's law to write the volume integral of \( \nabla \cdot j \) as a surface integral and assuming \( j(x) = 0 \) on that surface we find:

**Q is constant in time!**

using free field expansions, we get:

\[ \varphi(x) = \int \! dk \, [a(k)e^{ikx} + b^*(k)e^{-ikx}] \]

\[ \varphi^\dagger(x) = \int \! dk \, [b(k)e^{ikx} + a^*(k)e^{-ikx}] \]

for an interacting theory these formulas are valid at any given time

\[ Q = \int \! dk \, [a^*(k)a(k) - b(k)b^*(k)] \]

counts the number of \( a \) particles minus the number of \( b \) particles; it is time independent and so the scattering amplitudes do not change the value of \( Q \); in Feynman diagrams \( Q \) is conserved in every vertex.
Another use of Noether current:

Consider a transformation of fields that change the lagrangian density by a total divergence:

\[ \delta \mathcal{L}(x) = \partial_\mu K^\mu(x) \]

there is still a conserved current:

\[ j^\mu(x) = \frac{\partial \mathcal{L}(x)}{\partial (\partial_\mu \varphi_a(x))} \delta \varphi_a(x) - K^\mu(x) \]

**e.g.** space-time translations:

\[
\begin{align*}
\varphi_a(x) &\rightarrow \varphi_a(x-a) \\
\varphi_a(x) &\rightarrow \varphi_a(x) - a^\nu \partial_\nu \varphi_a(x) \\
\delta \varphi_a(x) &\rightarrow -a^\nu \partial_\nu \varphi_a(x)
\end{align*}
\]

we get:

\[
\begin{align*}
j^\mu(x) &= \frac{\partial \mathcal{L}(x)}{\partial (\partial_\mu \varphi_a(x))} (-a^\nu \partial_\nu \varphi_a(x)) + a^\mu \mathcal{L}(x) \\
&= a_\nu T^{\mu\nu}(x) \\
T^{\mu\nu}(x) &\equiv -\frac{\partial \mathcal{L}(x)}{\partial (\partial_\mu \varphi_a(x))} \partial_\nu \varphi_a(x) + g^{\mu\nu} \mathcal{L}(x)
\end{align*}
\]

stress-energy or energy-momentum tensor
for a theory of a set of real scalar fields:

\[ \mathcal{L} = -\frac{1}{2} \partial^\mu \varphi_a \partial_\mu \varphi_a - V(\varphi) \]

we get:

\[ T^{\mu\nu}(x) = -\frac{\partial \mathcal{L}(x)}{\partial (\partial_\mu \varphi_a(x))} \partial^\nu \varphi_a(x) + g^{\mu\nu} \mathcal{L}(x) \]

\[ T^{\mu\nu} = \partial^\mu \varphi_a \partial^\nu \varphi_a + g^{\mu\nu} \mathcal{L} \]

in particular:

\[ T^{00} = \frac{1}{2} \Pi_a^2 + \frac{1}{2} (\nabla \varphi_a)^2 + V(\varphi) \]

\[ \Pi_a = \partial_0 \varphi_a \]

then by Lorentz symmetry the momentum density must be:

\[ T^{0j} = \partial^0 \varphi_a \partial^j \varphi_a = -\Pi_a \nabla^j \varphi_a \]

plugging in the field expansions, we get:

\[ P^j = \int d^3x T^{0j}(x) = \int \frac{dk}{(2\pi)^3} \, k^j \, a^\dagger_a(k)a_a(k) \]

as expected
The energy-momentum four-vector is:

\[ P^\mu = \int d^3x \ T^{0\mu}(x) \]

Recall, we defined the space-time translation operator

\[ T(a) \equiv \exp(-iP^\mu a_\mu) \]

so that

\[ T(a)^{-1} \varphi_a(x) T(a) = \varphi_a(x - a) \]

we can easily verify it; for an infinitesimal transformation it becomes:

\[ [\varphi_a(x), P^\mu] = \frac{1}{i} \partial^\mu \varphi_a(x) \]

it is straightforward to verify this by using the canonical commutation relations for \( \varphi_a(x) \) and \( \Pi_a(x) \).
The same procedure can be repeated for Lorentz transformations:

\[ \varphi_a(x) \rightarrow \varphi_a(x + \delta \omega \cdot x) \]

the resulting conserved current is:

\[
\mathcal{M}^{\mu \nu \rho}(x) = x^\nu T^{\mu \rho}(x) - x^\rho T^{\mu \nu}(x)
\]

antisymmetric in the last two indices as a result of \( \delta \omega^{\nu \rho} \) being antisymmetric

the conserved charges associated with this current are:

\[
M^{\nu \rho} = \int d^3x \, M^{0 \nu \rho}(x)
\]

generators of the Lorentz group

again, one can check all the commutators...
Discrete symmetries: P, T, C and Z

Recall from S-2:

Infinitesimal Lorentz transformation:

\[ \Lambda_{\mu}^{\nu} = \delta_{\mu}^{\nu} + \delta \omega_{\mu}^{\nu} \]

not all LT can be obtained by compounding ILTs!

\[ (\Lambda^{-1})^{\rho}_{\nu} = \Lambda_{\nu}^{\rho} \rightarrow (\det \Lambda)^{-1} = \det \Lambda \]

\[ \det \Lambda = \pm 1 \]

+1 proper

-1 improper

proper LTs form a subgroup of Lorentz group; ILTs are proper!

Another subgroup - orthochronous LTs,

\[ g_{\mu \nu} \Lambda_{\rho}^{\mu} \Lambda_{\sigma}^{\nu} = g_{\rho \sigma} \rightarrow (\Lambda_{0}^{0})^{2} - \Lambda_{i}^{i} \Lambda_{i}^{0} = 1 \]

\[ \Lambda_{0}^{0} \geq +1 \]

\[ \Lambda_{0}^{0} \leq -1 \]

ILT are orthochronous!
When we say theory is **Lorentz invariant** we mean it is invariant under **proper orthochronous** subgroup only (those that can be obtained by compounding ILTs)

Transformations that take us out of **proper orthochronous** subgroup are **parity** and **time reversal**:

$$\mathcal{P}^\mu_\nu = (\mathcal{P}^{-1})^\mu_\nu = \begin{pmatrix} +1 & -1 \\ -1 & -1 \end{pmatrix} \quad \text{orthochronous but improper}$$

$$\mathcal{T}^\mu_\nu = (\mathcal{T}^{-1})^\mu_\nu = \begin{pmatrix} -1 & +1 \\ +1 & +1 \end{pmatrix} \quad \text{nonorthochronous and improper}$$

A quantum field theory doesn’t have to be invariant under P or T.
For every proper orthochronous LT there is a unitary operator:

\[ U(\Lambda)^{-1} \varphi(x) U(\Lambda) = \varphi(\Lambda^{-1} x) \]

we expect the same for parity and time-reversal

\[ P \equiv U(\mathcal{P}) \]
\[ T \equiv U(\mathcal{T}) \]

so that

\[ P^{-1} \varphi(x) P = \varphi(\mathcal{P} x) \]
\[ T^{-1} \varphi(x) T = \varphi(T x) \]

since \( \mathcal{P}^2 = 1 \) and \( T^2 = 1 \) we need:

\[ P^{-2} \varphi(x) P^2 = \varphi(x) \]
\[ T^{-2} \varphi(x) T^2 = \varphi(x) \]

that can be also satisfied with:

\[ P^{-1} \varphi(x) P = -\varphi(\mathcal{P} x) \]
\[ T^{-1} \varphi(x) T = -\varphi(T x) \]
We can choose the transformation properties of fields. It is a part of specifying the theory. But if possible we want to have lagrangian density even under both parity and time-reversal, 

\[ P^{-1} \mathcal{L}(x) P = + \mathcal{L}(\mathcal{P}x) \]
\[ T^{-1} \mathcal{L}(x) T = + \mathcal{L}(\mathcal{T}x) \]

so that parity and time-reversal are conserved.

Note: time-reversal operator must be antiunitary: 

\[ T^{-1}iT = -i \]

to see it, let’s look at transformations of the energy-momentum 4-vector:

\[ P^{-1} P^\mu P = \mathcal{P}^\mu_\nu P^\nu \]
\[ T^{-1} P^\mu T = -\mathcal{T}^\mu_\nu P^\nu \]

can be checked directly using:

\[ P^\mu = \int d^3x \ T^{0\mu}(x) \]
\[ T^{00} = \frac{1}{2} \Pi_a^2 + \frac{1}{2} (\nabla \varphi_a)^2 + V(\varphi) \]
\[ T^{0j} = \partial^0 \varphi_a \partial^j \varphi_a = -\Pi_a \nabla^j \varphi_a \]

the same result for scalar and pseudoscalar
for $\mu = 0$ we have:

\[
P^{-1}H P = +H \quad \text{GOOD}
\]

\[
T^{-1}HT = +H
\]

if $T$ was unitary, $T^{-1}P^\mu T = T^\mu \nu P^\nu$ we would have $T^{-1}HT = -H$ which is a DISASTER since Hamiltonian is invariant under time-reversal only if $H = -H$ and so $H = 0$.

Let's trace the origin of antiunitarity:

the spacetime translation operator

\[
T(a) = \exp(-iP \cdot a)
\]

implies:

\[
U(\Lambda)^{-1}T(a)U(\Lambda) = T(\Lambda^{-1}a)
\]
\[ U(\Lambda)^{-1}T(a)U(\Lambda) = T(\Lambda^{-1}a) \]

\[ T(a) = \exp(-iP \cdot a) \]

for an infinitesimal translation we get:

\[ U(\Lambda)^{-1}(I - ia_\mu P^\mu)U(\Lambda) = I - i(\Lambda^{-1})_{\nu}^{\mu}a_\mu P^\nu \]

\[ = I - i\Lambda_{\nu}^{\mu}a_\mu P^\nu. \]

similarly for time-reversal:

\[ T^{-1}(I - ia_\mu P^\mu)T = I - iT_{\mu}^{\nu}a_\mu P^\nu \]

comparing linear terms in a we see that in order to get

\[ T^{-1}P^\mu T = -T_{\mu}^{\nu}P^\nu \]

we need

\[ T^{-1}iT = -i \]

T is antiunitary
$Z_2$ symmetry:

we want to consider a possibility that the sign of a scalar field changes under a symmetry transformation (that does not act on spacetime arguments). The corresponding unitary operator is:

$$Z^{-1} \varphi_a(x) Z = \eta_a \varphi_a(x)$$

$\eta_a = +1$ or $-1$  

$Z^{-1} = Z$

$Z^2 = 1$

$Z_2$ symmetry

e.g. a theory of a complex scalar field:

$$\varphi = (\varphi_1 + i\varphi_2)/\sqrt{2}$$

$$\mathcal{L} = -\partial^\mu \varphi^\dagger \partial_\mu \varphi - m^2 \varphi^\dagger \varphi - \frac{1}{4} \lambda (\varphi^\dagger \varphi)^2$$

$$= -\frac{1}{2} \partial^\mu \varphi_1 \partial_\mu \varphi_1 - \frac{1}{2} \partial^\mu \varphi_2 \partial_\mu \varphi_2 - \frac{1}{2} m^2 (\varphi_1^2 + \varphi_2^2) - \frac{1}{16} \lambda (\varphi_1^2 + \varphi_2^2)^2$$

has U(1) symmetry: $\varphi(x) \rightarrow e^{-i\alpha} \varphi(x)$, equivalent to SO(2): 

$$\begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} \rightarrow \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix}$$

but also an additional discrete symmetry:

$$\varphi(x) \leftrightarrow \varphi^\dagger(x)$$

$$\begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} \rightarrow \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix}$$

charge conjugation
Charge conjugation always occurs as a companion to a U(1) symmetry; it enlarges $SO(2)$ symmetry (the group of 2x2 orthogonal matrices with determinant +1) into $O(2)$ symmetry (the group of 2x2 orthogonal matrices)

$$
\begin{pmatrix}
\varphi_1(x) \\
\varphi_2(x)
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\cos \alpha & \sin \alpha \\
-\sin \alpha & \cos \alpha
\end{pmatrix}
\begin{pmatrix}
\varphi_1(x) \\
\varphi_2(x)
\end{pmatrix}
$$

$$
\begin{pmatrix}
\varphi_1(x) \\
\varphi_2(x)
\end{pmatrix}
\rightarrow
\begin{pmatrix}
+1 & 0 \\
0 & -1
\end{pmatrix}
\begin{pmatrix}
\varphi_1(x) \\
\varphi_2(x)
\end{pmatrix}
$$

We can define the operator of charge conjugation:

$$C^{-1} \varphi(x) C = \varphi^\dagger(x)$$

or

$$C^{-1} \varphi_1(x) C = +\varphi_1(x)$$

$$C^{-1} \varphi_2(x) C = -\varphi_2(x)$$

and the charge conjugation is a symmetry of the theory:

$$C^{-1} \mathcal{L}(x) C = \mathcal{L}(x)$$

Scattering amplitudes must be unchanged if we exchange all a-type particles (charge +1) with all b-type particles (charge -1). This is only possible if both particles have the same mass; we say particle b is antiparticle of a.
Another example of $\mathbb{Z}_2$ symmetry:

consider $\varphi^4$ theory:

$$\mathcal{L} = -\frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} m^2 \varphi^2 - \frac{1}{24} \lambda \varphi^4$$

this theory is obviously invariant under:

$$Z^{-1} \varphi(x) Z = -\varphi(x)$$

the ground state (if unique) must also be an eigenstate of $Z$; we can fix the phase of $Z$ via:

$$Z |0\rangle = Z^{-1} |0\rangle = + |0\rangle$$

any choice would be fine

and then we have:

$$\langle 0 | \varphi(x) | 0 \rangle = \langle 0 | ZZ^{-1} \varphi(x) ZZ^{-1} | 0 \rangle$$

$$= -\langle 0 | \varphi(x) | 0 \rangle .$$

the $\mathbb{Z}_2$ symmetry implies that there is no need for a counterterm!
Nonabelian symmetries

Let’s generalize the theory of two real scalar fields:

\[ \mathcal{L} = -\frac{1}{2} \partial^\mu \varphi_1 \partial_\mu \varphi_1 - \frac{1}{2} \partial^\mu \varphi_2 \partial_\mu \varphi_2 - \frac{1}{2} m^2 (\varphi_1^2 + \varphi_2^2) - \frac{1}{16} \lambda (\varphi_1^2 + \varphi_2^2)^2 \]

to the case of \( N \) real scalar fields:

\[ \mathcal{L} = -\frac{1}{2} \partial^\mu \varphi_i \partial_\mu \varphi_i - \frac{1}{2} m^2 \varphi_i \varphi_i - \frac{1}{16} \lambda (\varphi_i \varphi_i)^2 \]

the lagrangian is clearly invariant under the \( \text{SO}(N) \) transformation:

\[ \varphi_i(x) \rightarrow R_{ij} \varphi_j(x) \]

orthogonal matrix with \( \text{det} \ = \ 1 \)

\[ R^T = R^{-1} \]

\[ \text{det} \ R = +1 \]

lagrangian has also the \( \mathbb{Z}_2 \) symmetry, \( \varphi_i(x) \rightarrow -\varphi_i(x) \), that enlarges \( \text{SO}(N) \) to \( \text{O}(N) \)
infinitesimal $\text{SO}(N)$ transformation:

$$R_{ij} = \delta_{ij} + \theta_{ij} + O(\theta^2)$$

antisymmetric\hspace{1cm}$$R^T = R^{-1}$$

$$R^T_{ij} = \delta_{ij} + \theta_{ji}$$

real\hspace{1cm}$$R^{-1}_{ij} = \delta_{ij} - \theta_{ij}$$

$$\text{Im}(R^{-1}R)_{ij} = \text{Im} \sum_k R_{ki}R_{kj} = 0$$

$$(N^2 \text{ linear combinations of Im parts } = 0)$$

there are $\frac{1}{2}N(N-1)$ linearly independent real antisymmetric matrices, and we can write:

$$\theta_{jk} = -i \theta^a (T^a)_{jk}$$

or $R = e^{-i \theta^a T^a}$.

The commutator of two generators is a lin. comb. of generators:

$$[T^a, T^b] = i f^{abc} T^c$$

we choose normalization: $\text{Tr}(T^a T^b) = 2 \delta^{ab}$

structure constants of the $\text{SO}(N)$ group
e.g. $SO(3)$:

\begin{align*}
(T^a)_{ij} &= -i\epsilon^{aij} \\
[T^a, T^b] &= i\epsilon^{abc}T^c \\
\epsilon^{123} &= +1
\end{align*}

Levi-Civita symbol
consider now a theory of $N$ complex scalar fields:

$$
\mathcal{L} = -\partial^\mu \varphi_i^\dagger \partial_\mu \varphi_i - m^2 \varphi_i^\dagger \varphi_i - \frac{1}{4} \lambda (\varphi_i^\dagger \varphi_i)^2
$$

the lagrangian is clearly invariant under the $U(N)$ transformation:

$$
\varphi_i(x) \rightarrow U_{ij} \varphi_j(x)
$$

$$
U^\dagger = U^{-1}
$$

we can always write $U_{ij} = e^{-i\theta \tilde{U}_{ij}}$ so that $\det \tilde{U} = +1$.

actually, the lagrangian has larger symmetry, $SO(2N)$:

$$
\varphi_j = (\varphi_{j1} + i\varphi_{j2})/\sqrt{2}
$$

$$
\varphi_j^\dagger \varphi_j = \frac{1}{2} (\varphi_{11}^2 + \varphi_{12}^2 + \ldots + \varphi_{N1}^2 + \varphi_{N2}^2)
$$
infinitesimal $\text{SU}(N)$ transformation:

$$\tilde{U}_{ij} = \delta_{ij} - i\theta^a (T^a)_{ij} + O(\theta^2)$$

or $\tilde{U} = e^{-i\theta^a T^a}$.

there are $N^2 - 1$ linearly independent traceless hermitian matrices:

e.g. $\text{SU}(2)$ - 3 Pauli matrices

$\text{SU}(3)$ - 8 Gell-Mann matrices

the structure coefficients are $f^{abc} = 2\epsilon^{abc}$, the same as for $\text{SO}(3)$.