Continuous symmetries and conserved currents

Consider a set of scalar fields \( \varphi_a(x) \), and a lagrangian density

\[
\mathcal{L}(x) = \mathcal{L}(\varphi_a(x), \partial_\mu \varphi_a(x))
\]

let's make an infinitesimal change: \( \varphi_a(x) \rightarrow \varphi_a(x) + \delta \varphi_a(x) \)

\[
\mathcal{L}(x) \rightarrow \mathcal{L}(x) + \delta \mathcal{L}(x)
\]

\[
\delta \mathcal{L}(x) = \frac{\partial \mathcal{L}}{\partial \varphi_a(x)} \delta \varphi_a(x) + \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_a(x)} \partial_\mu \delta \varphi_a(x)
\]

variation of the action:

\[
\frac{\delta S}{\delta \varphi_a(x)} = \int d^4y \frac{\delta \mathcal{L}(y)}{\delta \varphi_a(x)}
\]

\[
= \int d^4y \left[ \frac{\partial \mathcal{L}(y)}{\partial \varphi_a(x)} \delta \varphi_a(x) + \frac{\partial \mathcal{L}(y)}{\partial \partial_\mu \varphi_a(x)} \partial_\mu \delta \varphi_a(x) \right]
\]

\[
= \int d^4y \left[ \frac{\partial \mathcal{L}(y)}{\partial \varphi_a(x)} \delta \varphi_a(x) - \partial_{\mu} \frac{\partial \mathcal{L}(x)}{\partial \varphi_a(x)} \delta \varphi_a(x) - \frac{\partial \mathcal{L}(x)}{\partial \varphi_a(x)} \partial_\mu \delta \varphi_a(x) + \frac{\partial \mathcal{L}(x)}{\partial \varphi_a(x)} \partial_\mu \delta \varphi_a(x) \right]
\]

\[
= \int d^4y \left[ \delta \mathcal{L}(x) - \delta \mathcal{L}(x) \partial_\mu \delta \varphi_a(x) \right]
\]

\[
\delta \mathcal{L}(x) = \frac{\partial \mathcal{L}}{\partial \varphi_a(x)} \delta \varphi_a(x) + \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_a(x)} \partial_\mu \delta \varphi_a(x)
\]

thus we find:

\[
\delta \mathcal{L}(x) = \partial_\mu \left( \frac{\partial \mathcal{L}(x)}{\partial \partial_\mu \varphi_a(x)} \delta \varphi_a(x) \right) + \frac{\delta S}{\delta \varphi_a(x)} \delta \varphi_a(x)
\]

\[
\mathbf{j}^\mu(x) = \frac{\partial \mathcal{L}(x)}{\partial \partial_\mu \varphi_a(x)} \delta \varphi_a(x)
\]

this is called Noether current; now we have:

\[
\partial_\mu \mathbf{j}^\mu(x) = 0 \text{ if eqs. of motion are satisfied}
\]

\[
\partial_\mu \mathbf{j}^\mu(x) = \delta \mathcal{L}(x) - \frac{\delta S}{\delta \varphi_a(x)} \delta \varphi_a(x)
\]

if a set of infinitesimal transformations leaves the lagrangian unchanged, invariant, \( \delta \mathcal{L} = 0 \), the Noether current is conserved!

\[
\partial_\mu \mathbf{j}^\mu = 0
\]

charge density

current density
we set
\[ \hbar = c = 1 \]

any quantity \( A \) has units of mass to some power
that we call \([A]\), e.g:
\[
\begin{align*}
[m] &= +1 \\
[x^\mu] &= -1 \\
[\partial^\mu] &= +1 \\
[d^d x] &= -d
\end{align*}
\]

it allows us to convert
a time \( T \) to a length \( L \):
\[ T = c^{-1} L \]

a length to an inverse mass:
\[ L = \hbar c^{-1} M^{-1} \]

in \( d \) spacetime dimensions

the action appears in the exponential and so
\[ [S] = 0 \]

and for the lagrangian density we have:
\[ [\mathcal{L}] = d \]

from the kinetic term:
\[
\mathcal{L} = -\frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{3} m^2 \varphi^2 + \frac{1}{5} g \varphi^3
\]

\[ [\varphi] = \frac{1}{2} (d - 2) \]

in 4 dimensions:
\[ [\varphi] = 1 \]

functional derivative:
\[
\frac{\delta}{\delta f(t_1)} f(t_2) = \delta(t_1 - t_2)
\]
similarly, in 4d:

\[
\frac{\delta \varphi_b(y)}{\delta \varphi_a(x)} = \delta_{ba} \delta^4(y-x)
\]

Noether current:

\[
\partial_\mu j_\mu(x) = \delta \mathcal{L}(x) - \frac{\delta S}{\delta \varphi_a(x)} \delta \varphi_a(x)
\]

and so on ...

Consider a theory of a complex scalar field:

\[
\mathcal{L} = -\partial^\mu \varphi^\dagger \partial_\mu \varphi - m^2 \varphi^\dagger \varphi - \frac{1}{4} \lambda (\varphi^\dagger \varphi)^2
\]

clearly \( \mathcal{L} \) is left invariant by:

\[
\varphi(x) \rightarrow e^{-i\alpha} \varphi(x) \quad \text{U(1) transformation}
\]

(transformation by a unitary 1x1 matrix)

in terms of two real scalar fields we get:

\[
\varphi = (\varphi_1 + i \varphi_2)/\sqrt{2}
\]

\[
\mathcal{L} = -\frac{1}{2} \partial^\mu \varphi_1 \partial_\mu \varphi_1 - \frac{1}{2} \partial^\mu \varphi_2 \partial_\mu \varphi_2 - \frac{1}{2} m^2 (\varphi_1^2 + \varphi_2^2) - \frac{1}{16} \lambda (\varphi_1^2 + \varphi_2^2)^2
\]

and the U(1) transformation above is equivalent to:

\[
\begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} \rightarrow \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} \quad \text{SO(2) transformation}
\]

(transformation by an orthogonal 2x2 matrix with determinant = +1)
infinitesimal form of $\varphi(x) \rightarrow e^{-i\alpha \varphi(x)}$ is:

$$\varphi(x) \rightarrow \varphi(x) - i\alpha \varphi(x) ,$$

$$\varphi^\dagger(x) \rightarrow \varphi^\dagger(x) + i\alpha \varphi^\dagger(x) ,$$

we treat $\varphi$ and $\varphi^\dagger$ as independent fields

and the current is:

$$\alpha j^\mu = \frac{\partial L}{\partial (\partial_\mu \varphi)} \delta \varphi + \frac{\partial L}{\partial (\partial_\mu \varphi^\dagger)} \delta \varphi^\dagger$$

$$= (-\partial_\mu \varphi^\dagger)(-i\alpha \varphi) + (\partial_\mu \varphi)(+i\alpha \varphi^\dagger)$$

$$= \alpha \text{Im}(\varphi^\dagger \partial_\mu \varphi) ,$$

$$A\partial_\mu B \equiv A\partial_\mu B - (\partial_\mu A)B$$

which is equivalent to

$$j^\mu = \text{Im}(\varphi^\dagger \partial_\mu \varphi)$$

repeating the same for the SO(2) transformation:

$$\delta \varphi_1 = +\alpha \varphi_2$$

$$\delta \varphi_2 = -\alpha \varphi_1$$

the Noether current is:

$$L = -\frac{1}{2} \partial_\mu \varphi_1 \partial_\mu \varphi_1 - \frac{1}{2} \partial_\mu \varphi_2 \partial_\mu \varphi_2 - \frac{1}{2} m^2 (\varphi_1^2 + \varphi_2^2) - \frac{1}{2} \lambda (\varphi_1^2 + \varphi_2^2)^2$$

$$j^\mu(x) \equiv \frac{\partial L}{\partial (\partial_\mu \varphi_a(x))} \delta \varphi_a(x)$$

$$= (-\partial_\mu \varphi_1)(+\alpha \varphi_2) + (\partial_\mu \varphi_2)(-\alpha \varphi_1)$$

$$= \alpha (\varphi_1 \partial_\mu \varphi_2) ,$$

$$\varphi = (\varphi_1 + i\varphi_2)/\sqrt{2}$$
Let’s define the Noether charge:

$$Q \equiv \int d^3x \, j^0(x) = \int d^3x \, \text{Im}(\phi^* \partial^\mu \phi)$$

integrating \( \frac{\partial}{\partial t} j^0(x) + \nabla \cdot j(x) = 0 \) over \( d^3x \),

using Gauss’s law to write the volume integral of \( \nabla \cdot j \) as a surface integral and assuming \( j(x) = 0 \) on that surface

we find:

**Q is constant in time!**

using free field expansions, we get:

\[
\phi(x) = \int \frac{dk}{(2\pi)^3} [a(k)e^{ikx} + b^*(k)e^{-ikx}]
\]

\[
\phi^*(x) = \int \frac{dk}{(2\pi)^3} [b(k)e^{ikx} + a^*(k)e^{-ikx}]
\]

for an interacting theory these formulas are valid at any given time

\[
Q = \int \frac{dk}{2} [a^*(k)a(k) - b(k)b^*(k)]
\]

counts the number of a particles minus the number of b particles; it is time independent and so the scattering amplitudes do not change the value of Q; in Feynman diagrams Q is conserved in every vertex.

Another use of Noether current:

Consider a transformation of fields that change the lagrangian density by a total divergence:

$$\delta \mathcal{L}(x) = \partial_\mu K^\mu(x)$$

there is still a conserved current:

$$j^\mu(x) = \frac{\partial \mathcal{L}(x)}{\partial (\partial_\mu \phi_a(x))} \delta \phi_a(x) - K^\mu(x)$$

e.g. space-time translations:

\[
\begin{align*}
\phi_a(x) &\rightarrow \phi_a(x-a) \\
\phi_a(x) &\rightarrow \phi_a(x) - a^\nu \partial_\nu \phi_a(x) \\
\delta \phi_a(x) &\rightarrow -a^\nu \partial_\nu \phi_a(x)
\end{align*}
\]

\[
\begin{align*}
\mathcal{L}(x) &\rightarrow \mathcal{L}(x-a) \\
\delta \mathcal{L}(x) &\rightarrow -a^\nu \partial_\nu \mathcal{L}(x) = -\partial_\nu (a^\nu \mathcal{L}(x)) \\
K^\nu(x) &\rightarrow -a^\nu \mathcal{L}(x)
\end{align*}
\]

we get:

$$j^\mu(x) = \frac{\partial \mathcal{L}(x)}{\partial (\partial_\mu \phi_a(x))} (-a^\nu \partial_\nu \phi_a(x)) + a^\mu \mathcal{L}(x)$$

$$= a_\nu T^{\mu\nu}(x)$$

stress-energy or energy-momentum tensor
for a theory of a set of real scalar fields:

$$\mathcal{L} = -\frac{1}{2} \partial^\mu \varphi_a \partial_\mu \varphi_a - V(\varphi)$$

we get:

$$T^{\mu\nu}(x) \equiv -\frac{\partial \mathcal{L}(x)}{\partial (\partial_\mu \varphi_a(x))} \partial_\nu \varphi_a(x) + g^{\mu\nu} \mathcal{L}(x)$$

$$T^{\mu\nu} = \partial^\mu \varphi_a \partial^\nu \varphi_a + g^{\mu\nu} \mathcal{L}$$

in particular:

$$T^{00} = \frac{1}{2} \Pi_a^2 + \frac{1}{2} (\nabla \varphi_a)^2 + V(\varphi)$$

Hamiltonian density $\mathcal{H}$

then by Lorentz symmetry the momentum density must be:

$$T^{0j} = \partial^0 \varphi_a \partial^j \varphi_a = -\Pi_a \nabla^j \varphi_a$$

plugging in the field expansions, we get:

$$P^j = \int d^3 x T^{0j}(x) = \int \frac{dk}{2\pi} k^j a_a^\dagger(k) a_a(k)$$

as expected.

The energy-momentum four-vector is:

$$P^\mu = \int d^3 x T^{0\mu}(x)$$

Recall, we defined the space-time translation operator

$$T(a) \equiv \exp(-i P^\mu a_\mu)$$

so that

$$T(a)^{-1} \varphi_a(x) T(a) = \varphi_a(x - a)$$

we can easily verify it; for an infinitesimal transformation it becomes:

$$[\varphi_a(x), P^\mu] = \frac{1}{i} \partial^\mu \varphi_a(x)$$

it is straightforward to verify this by using the canonical commutation relations for $\varphi_a(x)$ and $\Pi_a(x)$.
The same procedure can be repeated for Lorentz transformations:

$$\varphi_a(x) \rightarrow \varphi_a(x + \delta \omega \cdot x)$$

the resulting conserved current is:

$$\mathcal{M}^{\mu\nu\rho}(x) = x^\nu T^{\mu\rho}(x) - x^\rho T^{\mu\nu}(x)$$

antisymmetric in the last two indices as a result of $\delta \omega^{\nu\rho}$ being antisymmetric

the conserved charges associated with this current are:

$$M^{\nu\rho} = \int d^3x \mathcal{M}^{0\nu\rho}(x)$$

generators of the Lorentz group

again, one can check all the commutators...

\textbf{Discrete symmetries: P, T, C and Z}

Recall from S-2:

\textbf{Infinitesimal Lorentz transformation:}

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \delta \omega^\mu_\nu$$

not all LT can be obtained by compounding ILTs!

$$(\Lambda^{-1})^\rho_\nu = \Lambda^\rho_\nu$$

$$\det \Lambda = \pm 1$$

proper LTs form a subgroup of Lorentz group; ILTs are proper!

Another subgroup - orthochronous LTs, $\Lambda^0_0 \geq +1$

$$g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = g_{\rho\sigma}$$

$$g_{00} \Lambda^0_0 \geq 1$$

ILT are orthochronous!
When we say theory is Lorentz invariant we mean it is invariant under proper orthochronous subgroup only (those that can be obtained by compounding ILTs)

Transformations that take us out of proper orthochronous subgroup are parity and time reversal:

\[
\mathcal{P}^\mu_\nu = (\mathcal{P}^{-1})^\mu_\nu = \begin{pmatrix}
+1 \\
-1 \\
-1 \\
-1
\end{pmatrix}
\text{ orthochronous but improper}
\]

\[
\mathcal{T}^\mu_\nu = (\mathcal{T}^{-1})^\mu_\nu = \begin{pmatrix}
-1 \\
+1 \\
+1 \\
+1
\end{pmatrix}
\text{ nonorthochronous and improper}
\]

A quantum field theory doesn’t have to be invariant under P or T.

For every proper orthochronous LT there is a unitary operator:

\[
U(\Lambda)^{-1}\varphi(x)U(\Lambda) = \varphi(\Lambda^{-1}x)
\]

we expect the same for parity and time-reversal

\[
P \equiv U(\mathcal{P}) \\
T \equiv U(\mathcal{T})
\]

so that

\[
P^{-1}\varphi(x)P = \varphi(\mathcal{P}x) \quad \text{scalar (even under parity)}
\]

\[
T^{-1}\varphi(x)T = \varphi(\mathcal{T}x) \quad \text{pseudoscalar (odd under parity)}
\]

since \(\mathcal{P}^2 = 1\) and \(\mathcal{T}^2 = 1\) we need:

\[
P^{-2}\varphi(x)P^2 = \varphi(x)
\]

\[
T^{-2}\varphi(x)T^2 = \varphi(x)
\]

that can be also satisfied with:

\[
P^{-1}\varphi(x)P = -\varphi(\mathcal{P}x)
\]

\[
T^{-1}\varphi(x)T = -\varphi(\mathcal{T}x)
\]
We can choose the transformation properties of fields. It is a part of specifying the theory. But if possible we want to have lagrangian density even under both parity and time-reversal,

\[ P^{-1}L(x)P = +L(Px) \]
\[ T^{-1}L(x)T = +L(Tx) \]

so that parity and time-reversal are conserved.

Note: time-reversal operator must be antiunitary:

\[ T^{-1}iT = -i \]

to see it, let's look at transformations of the energy-momentum 4-vector:

\[ P^{-1}P^\mu P = P^\mu \]
\[ U^{-1}P^\mu U = U^\mu \]

\[ T^{-1}P^\mu T = -T^\mu \]

can be checked directly using:

\[ P^\mu = \int d^4x \, T^0\mu(x) \]

\[ T^{00} = \frac{1}{2} \Pi_a^2 + \frac{1}{2} (\nabla \varphi_a)^2 + V(\varphi) \]

For \( \mu = 0 \) we have:

\[ P^{-1}HP = +H \quad \text{GOOD} \]
\[ T^{-1}HT = +H \]

If \( T \) was unitary, \( T^{-1}P^\mu T = T^\mu \) we would have \( T^{-1}HT = -H \) which is a DISASTER since Hamiltonian is invariant under time-reversal only if \( H = -H \) and so \( H = 0 \).

Let's trace the origin of antiunitarity:

the spacetime translation operator

\[ T(a) = \exp(-iP \cdot a) \]

\[ T(a)^{-1}\varphi(x)T(a) = \varphi(x-a) \]

implies:

\[ U^{-1}T(a)U = T(\Lambda^{-1}a) \]
\[ U(\Lambda)^{-1} T(a) U(\Lambda) = T(\Lambda^{-1} a) \]
\[ T(a) = \exp(-iP \cdot a) \]

for an infinitesimal translation we get:

\[ U(\Lambda)^{-1} (I - ia_{\mu} P^{\mu}) U(\Lambda) = I - i(\Lambda^{-1})_{\nu}^{\mu} a_{\mu} P^{\nu} \]
\[ = I - i\Lambda^{\mu}_{\nu} a_{\mu} P^{\nu}. \]

similarly for time-reversal:

\[ T^{-1} (I - ia_{\mu} P^{\mu}) T = I - iT^{\mu}_{\nu} a_{\mu} P^{\nu} \]

comparing linear terms in \( a \) we see that in order to get

\[ T^{-1} P^{\mu} T = -T^{\mu}_{\nu} P^{\nu} \]

we need

\[ T^{-1} i T = -i \]

\( T \) is antiunitary

\( Z_{2} \) symmetry:

we want to consider a possibility that the sign of a scalar field changes under a symmetry transformation (that does not act on spacetime arguments). The corresponding unitary operator is:

\[ Z^{-1} \varphi_{a}(x) Z = \eta_{a} \varphi_{a}(x) \]
\[ \eta_{a} = +1 \text{ or } -1 \]
\[ Z^{-1} = Z \]
\[ Z^{2} = 1 \]

\( Z_{2} \) symmetry

e.g. a theory of a complex scalar field:
\[ \varphi = (\varphi_{1} + i\varphi_{2})/\sqrt{2} \]
\[ \mathcal{L} = -\partial^{\mu} \varphi^\dagger \partial_{\mu} \varphi - m^{2} \varphi^\dagger \varphi - \frac{i}{2} \lambda(\varphi^\dagger \varphi)^{2} \]
\[ = -\frac{i}{2} \partial^{\mu} \varphi_{1} \partial_{\mu} \varphi_{1} - \frac{i}{2} \partial^{\mu} \varphi_{2} \partial_{\mu} \varphi_{2} - \frac{1}{2} m^{2}(\varphi_{1}^{2} + \varphi_{2}^{2}) - \frac{i}{2} \lambda(\varphi_{1}^{2} + \varphi_{2}^{2})^{2} \]

has \( U(1) \) symmetry: \( \varphi(x) \rightarrow e^{-i\alpha} \varphi(x) \), equivalent to \( SO(2) \):
\[ \begin{pmatrix} \varphi_{1}(x) \\ \varphi_{2}(x) \end{pmatrix} \rightarrow \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \varphi_{1}(x) \\ \varphi_{2}(x) \end{pmatrix} \]

but also an additional discrete symmetry:
\[ \varphi(x) \leftrightarrow \varphi^\dagger(x) \]
\[ \begin{pmatrix} \varphi_{1}(x) \\ \varphi_{2}(x) \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \varphi_{1}(x) \\ \varphi_{2}(x) \end{pmatrix} \]
charge conjugation
Charge conjugation always occurs as a companion to a U(1) symmetry; it enlarges SO(2) symmetry (the group of 2x2 orthogonal matrices with determinant +1) into O(2) symmetry (the group of 2x2 orthogonal matrices)

\[
\begin{pmatrix}
\varphi_1(x) \\
\varphi_2(x)
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\cos \alpha & \sin \alpha \\
-\sin \alpha & \cos \alpha
\end{pmatrix}
\begin{pmatrix}
\varphi_1(x) \\
\varphi_2(x)
\end{pmatrix}
\quad \text{or} \quad
\begin{pmatrix}
\varphi_1(x) \\
\varphi_2(x)
\end{pmatrix}
\rightarrow
\begin{pmatrix}
+1 & 0 \\
0 & -1
\end{pmatrix}
\begin{pmatrix}
\varphi_1(x) \\
\varphi_2(x)
\end{pmatrix}
\]

We can define the operator of charge conjugation:

\[ C^{-1} \varphi(x) C = \varphi^\dagger (x) \]

and the charge conjugation is a symmetry of the theory:

\[ C^{-1} \mathcal{L}(x) C = \mathcal{L}(x) \]

Scattering amplitudes must be unchanged if we exchange all a-type particles (charge +1) with all b-type particles (charge -1). This is only possible if both particles have the same mass; we say particle b is antiparticle of a.

Another example of \( Z_2 \) symmetry:

consider \( \varphi^4 \) theory:

\[
\mathcal{L} = \frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} m^2 \varphi^2 - \frac{1}{24} \lambda \varphi^4
\]

this theory is obviously invariant under:

\[ Z^{-1} \varphi(x) Z = -\varphi(x) \]

\[ Z^{-1} \mathcal{L} Z = H \]

we can fix the phase of \( Z \) via:

\[ Z|0\rangle = Z^{-1} |0\rangle = +|0\rangle \quad \text{any choice would be fine} \]

and then we have:

\[ \langle 0|\varphi(x)|0\rangle = \langle 0|Z Z^{-1} \varphi(x) ZZ^{-1}|0\rangle \]

\[ = -\langle 0|\varphi(x)|0\rangle \]

\[ \langle 0|\varphi(x)|0\rangle = 0 \]

the \( Z_2 \) symmetry implies that there is no need for a counterterm!
Nonabelian symmetries

Let's generalize the theory of two real scalar fields:

\[ \mathcal{L} = -\frac{1}{2} \partial^\mu \varphi_1 \partial_\mu \varphi_1 - \frac{1}{2} \partial^\mu \varphi_2 \partial_\mu \varphi_2 - \frac{1}{2} m^2 (\varphi_1^2 + \varphi_2^2) - \frac{1}{16} \lambda (\varphi_1^2 + \varphi_2^2)^2 \]

to the case of \( N \) real scalar fields:

\[ \mathcal{L} = -\frac{1}{2} \partial^\mu \varphi_i \partial_\mu \varphi_i - \frac{1}{2} m^2 \varphi_i \varphi_i - \frac{1}{16} \lambda (\varphi_i \varphi_i)^2 \]

the lagrangian is clearly invariant under the \( \text{SO}(N) \) transformation:

\[ \varphi_i(x) \rightarrow R_{ij} \varphi_j(x) \]

\[ R^T = R^{-1} \]

\[ \det R = \pm 1 \]

lagrangian has also the \( \mathbb{Z}_2 \) symmetry, \( \varphi_i(x) \rightarrow -\varphi_i(x) \), that enlarges \( \text{SO}(N) \) to \( O(N) \)

infinitesimal \( \text{SO}(N) \) transformation:

\[ R_{ij} = \delta_{ij} + \theta_{ij} + O(\theta^2) \]

\[ R_{ij}^T = R_{ij}^{-1} \]

\[ \theta_{ij} = \delta_{ij} + \theta_{ij} \]

\[ R \]

\[ \text{Im}(R^{-1}R)_{ij} = \text{Im} \sum_k R_{ki}R_{kj} = 0 \]

there are \( \frac{1}{2} N(N-1) \) linearly independent real antisymmetric matrices, and we can write:

\[ \theta_{jk} = -i \theta^a (T^a)_{jk} \]

or \( R = e^{-i \theta^a T^a} \).

The commutator of two generators is a lin. comb. of generators:

\[ [T^a, T^b] = i f^{abc} T^c \]

we choose normalization:

\[ \text{Tr}(T^a T^b) = 2 \delta^{ab} \]

\[ f^{abc} = -\frac{1}{2} i \text{Tr}( [T^a, T^b] T^d ) \]

structure constants of the \( \text{SO}(N) \) group
e.g. $SO(3)$:

\[
(T^a)_{ij} = -i \varepsilon^{aij}
\]

\[
[T^a, T^b] = i \varepsilon^{abc} T^c
\]

$\varepsilon^{123} = +1$

Levi-Civita symbol

Consider now a theory of $N$ complex scalar fields:

\[
\mathcal{L} = -\partial^\mu \varphi_i^\dagger \partial_\mu \varphi_i - m^2 \varphi_i^\dagger \varphi_i - \frac{1}{4} \lambda (\varphi_i^\dagger \varphi_i)^2
\]

The lagrangian is clearly invariant under the $U(N)$ transformation:

\[
\varphi_i(x) \rightarrow U_{ij} \varphi_j(x)
\]

\[
U^\dagger = U^{-1}
\]

We can always write $U_{ij} = e^{-i\theta} \widetilde{U}_{ij}$ so that $\det \widetilde{U} = +1$.

Actually, the lagrangian has larger symmetry, $SO(2N)$:

\[
\varphi_j = (\varphi_{j1} + i \varphi_{j2})/\sqrt{2}
\]

\[
\varphi_j^\dagger \varphi_j = \frac{1}{2} (\varphi_{11}^2 + \varphi_{12}^2 + \ldots + \varphi_{N1}^2 + \varphi_{N2}^2)
\]
infinitesimal $SU(N)$ transformation:

$$\tilde{U}_{ij} = \delta_{ij} - i\theta^a (T^a)_{ij} + O(\theta^2)$$

or $\tilde{U} = e^{-i\theta^a T^a}$.

there are $N^2 - 1$ linearly independent traceless hermitian matrices:

The structure coefficients are $f^{abc} = 2\varepsilon^{abc}$, the same as for $SO(3)$

$SU(2)$ - 3 Pauli matrices

$SU(3)$ - 8 Gell-Mann matrices

$U^\dagger = U^{-1}$

$\det \tilde{U} = +1$

$\ln \det A = \text{Tr} \ln A$