Plan for the rest of the semester

\[ \varphi \rightarrow \psi \alpha \]

\[ \varphi(x) \rightarrow e^{-i\alpha} \varphi(x) \]

\[ \varphi(x) \rightarrow e^{-i\alpha(x)} \varphi(x) \]

Representations of Lorentz Group

We defined a unitary operator that implemented a Lorentz transformation on a scalar field:

\[ U(\Lambda) \varphi(x) U(\Lambda) = \varphi(\Lambda^{-1} x) \]

and then a derivative transformed as:

\[ U(\Lambda) \partial^\mu \varphi(x) U(\Lambda) = \Lambda^{\mu}_{\rho} \partial^\rho \varphi(\Lambda^{-1} x) \]

it suggests, we could define a vector field that would transform as:

\[ U(\Lambda)^{-1} A^\mu(x) U(\Lambda) = \Lambda^{\mu}_{\rho} A^\rho(\Lambda^{-1} x) \]

and a tensor field \( B^{\mu\nu}(x) \) that would transform as:

\[ U(\Lambda)^{-1} B^{\mu\nu}(x) U(\Lambda) = \Lambda^{\mu}_{\rho} \Lambda^{\nu}_{\sigma} B^{\rho\sigma}(\Lambda^{-1} x) \]

for symmetric \( B^{\mu\nu}(x) = B^{\nu\mu}(x) \) and antisymmetric \( B^{\mu\nu}(x) = -B^{\nu\mu}(x) \) tensors, the symmetry is preserved by Lorentz transformations.

In addition, the trace \( T(x) \equiv g_{\mu\nu} B^{\mu\nu}(x) \) transforms as a scalar:

\[ U(\Lambda)^{-1} T(x) U(\Lambda) = T(\Lambda^{-1} x) \]

Thus a general tensor field can be written as:

\[ B^{\mu\nu}(x) = A^{\mu\nu}(x) + S^{\mu\nu}(x) + \frac{1}{2} g^{\mu\nu} T(x) \]

where different parts do not mix with each other under LT!

How do we find the smallest (irreducible) representations of the Lorentz group for a field with \( n \) vector indices?

Let's start with a field carrying a generic Lorentz index:

\[ U(\Lambda)^{-1} \varphi_A(x) U(\Lambda) = L_A^B(\Lambda) \varphi_B(\Lambda^{-1} x) \]

matrices that depend on \( \Lambda \),

they must obey the group composition rule

\[ L_A^B(\Lambda') L_B^C(\Lambda) = L_A^C(\Lambda' \Lambda) \]

we say these matrices form a representation of the Lorentz group.
For an infinitesimal transformation we had:

\[ U(1 + \delta \omega) = I + \frac{1}{2} \delta \omega_{\mu \nu} M^{\mu \nu} \]

where the generators of the Lorentz group satisfied:

\[ [M^{\mu \nu}, M^{\rho \sigma}] = i \left( g^{\mu \rho} M^{\nu \sigma} - (\mu \leftrightarrow \nu) \right) - (\rho \leftrightarrow \sigma) \]

**Lie algebra of the Lorentz group**

or in components (angular momentum and boost),

\[ J_i \equiv \frac{1}{2} \varepsilon_{ijk} M^{jk} \]

\[ K_i \equiv M^{i0} \]

we have found:

\[ [J_i, J_j] = i \hbar \varepsilon_{ijk} J_k , \]
\[ [J_i, K_j] = i \hbar \varepsilon_{ijk} K_k , \]
\[ [K_i, K_j] = -i \hbar \varepsilon_{ijk} J_k \]

How do we find all possible sets of matrices that satisfy \[ M^{\mu \nu} = \delta^{\mu \nu} + \delta \omega^{\mu \nu} \]?

\[ [M^{\mu \nu}, M^{\rho \sigma}] = i \left( g^{\mu \rho} M^{\nu \sigma} - (\mu \leftrightarrow \nu) \right) - (\rho \leftrightarrow \sigma) \]

\[ \{J_i, J_j\} = i \hbar \varepsilon_{ijk} J_k , \]
\[ \{J_i, K_j\} = i \hbar \varepsilon_{ijk} K_k , \]
\[ \{K_i, K_j\} = -i \hbar \varepsilon_{ijk} J_k \]

the first one is just the usual set of commutation relations for angular momentum in QM:

for given \( j, 0, 1/2, 1, ... \) we can find three \( (2j+1) \times (2j+1) \) hermitian matrices \( J_1, J_2 \) and \( J_3 \) that satisfy the commutation relations and the eigenvalues of \( J_3 \) are \( -j, -j+1, ... , j \). such matrices constitute all of the inequivalent, irreducible representations of the Lie algebra of \( SO(3) \) equivalent to the Lie algebra of \( SU(2) \).

**Crucial observation:**

\[ \{J_i, J_j\} = i \hbar \varepsilon_{ijk} J_k , \]
\[ \{J_i, K_j\} = i \hbar \varepsilon_{ijk} K_k , \]
\[ \{K_i, K_j\} = -i \hbar \varepsilon_{ijk} J_k \]

The Lie algebra of the Lorentz group splits into two different \( SU(2) \) Lie algebras that are related by hermitian conjugation!

A representation of the Lie algebra of the Lorentz group can be specified by two integers or half-integers:

\( (2n+1, 2n'+1) \)

there are \( (2n+1)(2n'+1) \) different components of a representation they can be labeled by their angular momentum representations: since \( J_i = N_i + N_i' \) for given \( n \) and \( n' \) the allowed values of \( j \) are \( |n-n'|, |n-n'|+1, ... , n+n' \) (the standard way to add angular momenta, each value appears exactly once)
The simplest representations of the Lie algebra of the Lorentz group are:

\[(2n+1, 2n'+1)\]

\[(1, 1) = \text{scalar or singlet}\]
\[(2, 1) = \text{left-handed spinor}\]
\[(1, 2) = \text{right-handed spinor}\]
\[(2, 2) = \text{vector}\]

\[j = 0 \text{ and } 1\]

Left- and Right-handed spinor fields

Based on S-34

Let's start with a left-handed spinor field (left-handed Weyl field) \(\psi_a(x)\):

under Lorentz transformation we have:

\[U(\Lambda)^{-1}\psi_a(x) = L_a^b(\Lambda)\psi_b(\Lambda^{-1}x)\]

so that for \(i=1\) and \(j=2\):

\[(S_L^{ij})_a^b = \frac{1}{2} e^{ijk}\sigma_k\]

\[\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\]

Using

\[U(1+\delta\omega) = I + \frac{1}{2}\delta\omega_{\mu\nu}M^{\mu\nu}\]

we get

\[U(\Lambda)^{-1}\psi_a(x)U(\Lambda) = L_a^b(\Lambda)\psi_b(\Lambda^{-1}x)\]

\[\left[\psi_a(x), M^{\mu\nu}\right] = L^{\mu\nu}\psi_a(x) + (S_L^{\mu\nu})_a^b\psi_b(x)\]

\[L^{\mu\nu} = \frac{1}{2}(x^\mu\partial^\nu - x^\nu\partial^\mu)\]

present also for a scalar field to simplify the formulas, we can evaluate everything at space-time origin, \(x^\mu = 0\)

and since \(M_{ij} = e^{ijk}J_k\), we have:

\[\varepsilon^{ijk}[\psi_a(0), J_k] = (S_L^{ij})_a^b\psi_b(0)\]

Once we set the representation matrices for the angular momentum operator, those for boosts \(J_k = M^{k0}\) follow from:

\[N_i \equiv \frac{1}{2}(J_i - iK_i)\]
\[N_i \equiv \frac{1}{2}(J_i + iK_i)\]

\[K_k = i(N_k - N_k)\]

\(N_i\) do not contribute when acting on a field in (2,1) representation and so the representation matrices for \(K_k\) are \(i\) times those for \(J_k\):

\[(S_L^{k0})_a^b = \frac{1}{2} i\sigma_k\]

\[(S_L^{ij})_a^b = \frac{1}{2} e^{ijk}\sigma_k\]
Let’s consider now a hermitian conjugate of a left-handed spinor field $\psi_a(x)$ (a hermitian conjugate of a (2,1) field should be a field in the (1,2) representation) = right-handed spinor field (right-handed Weyl field)

$$\left[ \psi_a(x) \right]^\dagger = \psi^\dagger_a(x)$$

we use dotted indices to distinguish (2,1) from (1,2)!

under Lorentz transformation we have:

$$U(\Lambda)^{-1} \psi^\dagger_a(x) U(\Lambda) = R_{a}^{\ d}(\Lambda) \psi^\dagger_d(\Lambda^{-1} x)$$

matrices in the (1,2) representation, that satisfy the group composition rule:

$$R_{a}^{\ b}(\Lambda') R_{b}^{\ c}(\Lambda) = R_{a}^{\ c}(\Lambda')$$

For an infinitesimal transformation we have:

$$R_{a}^{\ b}(1 + \omega) = \delta_{a}^{\ b} + \frac{i}{2} \delta \omega_{\mu\nu} (S^\mu_{\nu})_{a}^{\ b}$$

$$\left( S^\mu_{\nu} \right)_{a}^{\ b} = -\left( S^\mu_{\nu} \right)_{b}^{\ a}$$

in the same way as for the left-handed field we find:

$$\left[ \psi_a(x), M^{\mu\nu} \right] = L^{\mu\nu} \psi_a(x) + \left( S^\mu_{\nu} \right)_{a}^{\ b} \psi_b(x)$$

and we find:

$$\left[ \psi^\dagger_a(0), M^{\mu\nu} \right] = \left( S^\mu_{\nu} \right)_{a}^{\ b} \psi^\dagger_b(0)$$

taking the hermitian conjugate,

$$[M^{\mu\nu}, \psi_a(0)] = \left( S^\mu_{\nu} \right)_{a}^{\ b} \psi^\dagger_b(0)$$

we find:

$$\left( S^\mu_{\nu} \right)_{a}^{\ b} = -\left[ (S^\mu_{\nu})^{a}_{\ b} \right]^*$$

Let’s consider now a field that carries two (2,1) indices.

Under Lorentz transformation we have:

$$U(\Lambda)^{-1} C_{ab}(x) U(\Lambda) = L_{a}^{\ c}(\Lambda) L_{b}^{\ d}(\Lambda) C_{cd}(\Lambda^{-1} x)$$

Can we group 4 components of $C$ into smaller sets that do not mix under Lorentz transformation?

Recall from QM that two spin 1/2 particles can be in a state of total spin 0 or 1:

$$2 \otimes 2 = 1_A \oplus 3_S$$

1 antisymmetric spin 0 state

3 symmetric spin 1 states

Thus for the Lorentz group we have:

$$(2, 1) \otimes (2, 1) = (1, 1)_A \oplus (3, 1)_S$$

and we should be able to write:

$$C_{ab}(x) = \epsilon_{ab} D(x) + G_{ab}(x) \quad \epsilon_{ab} = -\epsilon_{ba}$$

$$G_{ab}(x) = G_{ba}(x)$$

$$D$$ is a scalar

$$U(\Lambda)^{-1} C_{ab}(x) U(\Lambda) = L_{a}^{\ c}(\Lambda) L_{b}^{\ d}(\Lambda) C_{cd}(\Lambda^{-1} x)$$

similar to

$$\Lambda_{\mu}^{\ a} \Lambda_{\nu}^{\ b} g_{\rho\sigma} = g_{\mu\nu}$$

is an invariant symbol of the Lorentz group (does not change under a Lorentz transformation that acts on all of its indices)

We can use it, and its inverse to raise and lower left-handed spinor indices:

$$\epsilon^{12} = \epsilon_{21} = +1, \quad \epsilon^{21} = \epsilon_{12} = -1 \quad \epsilon_{ab} \epsilon^{bc} = \delta^{ac}\delta_{ba}, \quad \epsilon_{ab} \epsilon^{bc} = \delta^{ac}\delta_{ba}$$

to raise and lower left-handed spinor indices:

$$\psi^\dagger_a(x) \equiv \epsilon^{ab} \psi_b(x)$$
\[ \varepsilon_{ab} = \delta_a^c, \quad \varepsilon^{ab} = \delta^c_b \]

\[ \psi^a(x) \equiv \varepsilon^{ab} \psi_b(x) \]

We also have:

\[ \psi_a = \varepsilon_{ab} \psi_b = \varepsilon_{ab} \varepsilon^{bc} \psi_c = \delta^{ac} \psi_c \]

we have to be careful with the minus sign, e.g.:

\[ \psi^a = \varepsilon^{ab} \psi_b = -\varepsilon^{ba} \psi_b = -\psi_b \varepsilon^{ba} = \psi_b \varepsilon^{ab} \]

or when contracting indices:

\[ \psi^a \chi_a = \varepsilon^{ab} \psi_b \chi_a = -\varepsilon^{ba} \psi_b \chi_a = -\psi_b \chi^b \]

Exactly the same discussion applies to two (1,2) indices:

\[ (1,2) \otimes (1,2) = (1,1)_A \oplus (1,3)_A \oplus (3,1)_A \oplus (3,3)_S \]

with \( \varepsilon_{,ab} \) defined in the same way as \( \varepsilon_{ab} : \varepsilon_{,ab} = -\varepsilon_{,ba} \)

Finally, let's consider a field that carries one undotted and one dotted index; it is in the (2,2) representation (vector):

\[ A_{\dot{a}a}(x) = \sigma_{\dot{a}a}^\mu A_\mu(x) \]

\[ \text{dictionary between the two notations} \]
\[ \text{it is an invariant symbol, we can deduce its existence from} \]
\[ (2,1) \otimes (1,2) \otimes (2,2) = (1,1) \otimes \ldots \]

A consistent choice with what we have already set for \( S_L^{\mu\nu} \) and \( S_R^{\mu\nu} \)

\[ \sigma_{\dot{a}a}^\mu = (I, \bar{\sigma}) \]

In general, whenever the product of a set of representations includes the singlet, there is a corresponding invariant symbol,

e.g. the existence of \( g_{\mu\nu} = g_{\nu\mu} \) follows from

\[ (2,2) \otimes (2,2) = (1,1)_S \oplus (1,3)_A \oplus (3,1)_A \oplus (3,3)_S \]

another invariant symbol we will use is completely antisymmetric Levi-Civita symbol:

\[ \varepsilon^{\mu\nu\rho\sigma} \]

\[ \varepsilon^{0123} = +1 \]

\[ \Lambda^\mu_{\alpha\beta} \Lambda^\nu_{\gamma\delta} \varepsilon^{\alpha\beta\gamma\delta} \]

is antisymmetric on exchange of any two of its uncontracted indices, and therefore must be proportional to \( \varepsilon^{\mu\nu\rho\sigma} \), the constant of proportionality is \( \det \Lambda \) which is +1 for proper Lorentz transformations.

Comparing the formula for a general field with two vector indices

\[ B^{\mu\nu}(x) = A^{\mu\nu}(x) + S^{\mu\nu}(x) + \frac{1}{4} g^{\mu\nu} T(x) \]

with

\[ (2,2) \otimes (2,2) = (1,1)_S \oplus (1,3)_A \oplus (3,1)_A \oplus (3,3)_S \]

we see that \( A \) is not irreducible and, since \( (3,1) \) corresponds to a symmetric part of undotted indices,

\[ \sigma_{ab} = \varepsilon_{ab} D(x) + C_{ab}(x) \]

we should be able to write it in terms of \( G \) and its hermitian conjugate.

see Srednicki
Fun with spinor indices

based on S-35

invariant symbol for raising and lowering spinor indices:

$$\epsilon^{12} = \epsilon^{21} = \epsilon_{21} = +1, \quad \epsilon^{21} = \epsilon^{12} = \epsilon_{12} = -1$$

$$\epsilon^{ab} = -\epsilon_{ab} = i \sigma_2$$

another invariant symbol:

$$\sigma^\mu_{a\dot{a}} = (I, \sigma)$$

$$A_{a\dot{a}}(x) = \sigma^\mu_{a\dot{a}} A_\mu(x)$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Simple identities:

$$\sigma^\mu_{a\dot{a}} \sigma_{\mu\dot{b}} = -2 \epsilon_{ab} \epsilon_{\dot{a}\dot{b}}$$

$$\epsilon^{ab} \epsilon^{\dot{a}\dot{b}} \sigma^\mu_{a\dot{a}} \sigma^{\nu}_{\dot{b}\dot{b}} = -2 g^{\mu\nu}$$

proportionality constants

from direct calculation

What can we learn about the generator matrices \((S^{\mu\nu}_{L})^b\) from invariant symbols?

\[ \text{from } \quad \epsilon^{ab} = L(\Lambda)^a L(\Lambda)^b \epsilon_{cd} : \]

for an infinitesimal transformation we had:

$$L^b_a (1 + \delta \omega) = \delta^b_a + \frac{1}{2} \delta \omega_{\mu\nu} (S^{\mu\nu}_{L})^b_a$$

and we find:

$$\epsilon^{ab} = \epsilon_{ab} + \frac{1}{2} \delta \omega_{\mu\nu} \left[ (S^{\mu\nu}_{L})^a_b \epsilon^{cb} + (S^{\mu\nu}_{L})^d_b \epsilon_{ad} \right] + O(\delta \omega^2)$$

$$= \epsilon_{ab} + \frac{1}{2} \delta \omega_{\mu\nu} \left[ -(S^{\mu\nu}_{L})_{ab} + (S^{\mu\nu}_{L})_{ba} \right] + O(\delta \omega^2).$$

similarly:

\[(S^{\mu\nu}_{L})^a_b = (S^{\mu\nu}_{L})^b_a\]

\[(S^{\mu\nu}_{R})^a_b = (S^{\mu\nu}_{R})^b_a\]

\[\text{from } \quad \sigma^{\mu}_{a\dot{a}} = \Lambda^0_{a\dot{a}} L(\Lambda)^a L(\Lambda)^b \sigma^{\nu}_{b\dot{b}} : \]

for infinitesimal transformations we had:

$$L^b_a (1 + \delta \omega) = \delta^b_a + \frac{1}{2} \delta \omega_{\mu\nu} (S^{\mu\nu}_{L})^b_a$$

$$L^b_a (1 + \delta \omega) = \delta^b_a + \frac{1}{2} \delta \omega_{\mu\nu} (S^{\mu\nu}_{L})^b_a$$

$$R^b_a (1 + \delta \omega) = \delta^b_a + \frac{1}{2} \delta \omega_{\mu\nu} (S^{\mu\nu}_{R})^b_a$$

isolating linear terms in \(\delta \omega_{\mu\nu}\) we have:

$$(g^{\mu\nu} \partial^\sigma - g^{\nu\sigma} \partial^\mu) \sigma^\nu_{a\dot{a}} + i (S^{\mu\nu}_{L})^b_a \sigma^\mu_{b\dot{b}} + i (S^{\mu\nu}_{R})^b_a \sigma^\mu_{b\dot{b}} = 0$$

multiplying by \(\sigma_{\rho\dot{c}\dot{c}}\) we have:

$$\sigma^\mu_{c\dot{c}} \sigma^\nu_{a\dot{a}} - \sigma^\nu_{c\dot{c}} \sigma^\mu_{a\dot{a}} + i (S^{\mu\nu}_{L})^b_a \sigma^\mu_{b\dot{b}} \sigma_{\rho\dot{c}\dot{c}} + i (S^{\mu\nu}_{R})^b_a \sigma^\mu_{b\dot{b}} \sigma_{\rho\dot{c}\dot{c}} = 0$$

multiplying by \(\epsilon^{ab}\) we get:

$$\epsilon^{ab} (S^{\mu\nu}_{L})^a_b = 0$$

$$\sigma^\mu_{c\dot{c}} \sigma^\nu_{a\dot{a}} - \sigma^\nu_{c\dot{c}} \sigma^\mu_{a\dot{a}} + 2 i (S^{\mu\nu}_{L})^b_a \epsilon_{ab} + 2 i (S^{\mu\nu}_{R})^b_a \epsilon_{ab} = 0$$

multiplying by \(\epsilon^{ac}\) we get:

$$\epsilon^{ac} (S^{\mu\nu}_{L})^a_c = 0$$

$$\sigma^\mu_{c\dot{c}} \sigma^\nu_{a\dot{a}} - \sigma^\nu_{c\dot{c}} \sigma^\mu_{a\dot{a}} + 2 i (S^{\mu\nu}_{L})^b_a \epsilon_{ac} + 2 i (S^{\mu\nu}_{R})^b_a \epsilon_{ac} = 0$$

let's define:

$$\tilde{\sigma}_{a\dot{a}} \equiv \epsilon^{ab} \epsilon^{\dot{a}\dot{b}} \sigma^\mu_{b\dot{b}}$$

$$\tilde{\sigma}_{a\dot{a}} = (I, \sigma)$$

$$\tilde{\sigma}_{a\dot{a}} = (I, -\sigma)$$

we find:

$$\sigma^{\mu}_{a\dot{a}} = \frac{1}{2} (\sigma^\mu_{a\dot{a}} \sigma^{\nu}_{\dot{b}\dot{b}} \sigma^{\nu}_{\dot{b}\dot{b}} + \sigma^{\nu}_{\dot{b}\dot{b}} \sigma^{\nu}_{\dot{b}\dot{b}})$$

$$\sigma_{a\dot{a}} = \frac{1}{2} (\sigma^\mu_{a\dot{a}} \sigma^{\nu}_{\dot{b}\dot{b}} \sigma^{\nu}_{\dot{b}\dot{b}} - \sigma^{\nu}_{\dot{b}\dot{b}} \sigma^{\nu}_{\dot{b}\dot{b}})$$

consistent with our previous choice! (homework)
Convention:
missing pair of contracted indices is understood to be written as:

\[ \chi \psi = \chi^a \psi_a \quad \text{and} \quad \chi^\dagger \psi^\dagger = \chi^\dagger_a \psi^\dagger_{\dot{a}} \]

thus, for left-handed Weyl fields we have:

\[ \chi \psi = \chi^a \psi_a \quad \text{and} \quad \chi^\dagger \psi^\dagger = \chi^\dagger_a \psi^\dagger_{\dot{a}} \]

spin 1/2 particles are fermions that anticommute: 
the spin-statistics theorem (later)

\[ \chi_a(x) \psi_b(y) = -\psi_b(y) \chi_a(x) \]

and we find:

\[ \chi \psi = \chi^a \psi_a = -\psi_a \chi^a = \psi^a \chi_a = \psi \chi \]

for hermitian conjugate we find:

\[ (\chi \psi)^\dagger = (\chi^a \psi_a)^\dagger = (\psi_a)^\dagger (\chi^a)^\dagger = \psi^\dagger_a \chi^\dagger_{\dot{a}} = \psi^\dagger \chi^\dagger \]

as expected if we ignored indices

and similarly:

\[ \psi^\dagger \chi^\dagger = \chi^\dagger \psi^\dagger \]

we will write a right-handed field always with a dagger!

Let’s look at something more complicated:

\[ \psi^\dagger \tilde{\sigma}^\mu \chi = \psi^\dagger_a \tilde{\sigma}^{\mu \dot{a} \dot{c}} \chi^\dagger_\dot{c} \]

it behaves like a vector field under Lorentz transformations:

\[ U(\Lambda)^{-1} [\psi^\dagger \tilde{\sigma}^\mu \chi] U(\Lambda) = \Lambda^\mu_\nu [\psi^\dagger \tilde{\sigma}^\nu \chi] \]

the hermitian conjugate is:

\[ [\psi^\dagger \tilde{\sigma}^\mu \chi]^\dagger = [\psi^\dagger_a \tilde{\sigma}^{\mu \dot{a} \dot{c}} \chi^\dagger_\dot{c}]^\dagger = \chi^\dagger_c (\tilde{\sigma}^{\mu \dot{a} \dot{c}})^* \psi^\dagger_a \]

\[ \tilde{\sigma}^\mu = (I - \tilde{\sigma}) \text{ is hermitian} \]

\[ = \chi^\dagger_c \tilde{\sigma}^{\mu \dot{a} \dot{c}} \psi^\dagger_a \]

\[ = \chi^\dagger_{\dot{c}} \tilde{\sigma}^{\mu \dot{a} \dot{c}} \psi^\dagger_a \]

\[ = \chi^\dagger_{\dot{c}} \tilde{\sigma}^{\mu^\dagger \dot{a}} \psi^\dagger_a \]