Lagrangians for spinor fields

Based on S-36, we want to find a suitable lagrangian for left- and right-handed spinor fields. It should be:

- Lorentz invariant and hermitian
- Quadratic in $\psi_a$ and $\psi_a^\dagger$

Equations of motion will be linear with plane wave solutions (suitable for describing free particles)

Terms with no derivative:
$$\psi \psi = \psi^a \psi_a = \varepsilon^{ab} \psi_b \psi_a$$

Terms with derivatives:
$$\partial^\mu \psi \partial_\mu \psi$$

would lead to a hamiltonian unbounded from below

to get a bounded hamiltonian the kinetic term has to contain both $\psi_a$ and $\psi_a^\dagger$, a candidate is:

$$i \psi^\dagger \sigma^\mu \partial_\mu \psi$$

is hermitian up to a total divergence

$$\left(i \psi^\dagger \sigma^\mu \partial_\mu \psi\right)^\dagger = \left(i \psi_a^\dagger \sigma^{\mu a c} \partial_\mu \psi_c\right)^\dagger$$

$$= -i \partial_\mu \psi_c^\dagger \left(\sigma^{\mu a c}\right)^* \psi_a$$

$$= -i \partial_\mu \psi_c^\dagger \bar{\sigma}^{\mu a c} \psi_a$$

$$= i \psi_c^\dagger \bar{\sigma}^{\mu a c} \partial_\mu \psi_a - i \partial_\mu \left(\psi_c^\dagger \bar{\sigma}^{\mu a c} \psi_a\right)$$

$$= i \psi_c^\dagger \bar{\sigma}^{\mu a c} \partial_\mu \psi_a$$

$$= i \psi_c^\dagger \bar{\sigma}^{\mu a c} \partial_\mu \psi - i \partial_\mu \left(\psi_c^\dagger \bar{\sigma}^{\mu a c} \psi\right).$$

$\sigma^{\mu a c} = (I, -\bar{\sigma})$

are hermitian

do not contribute to the action

The phase of $m$ can be absorbed into the definition of fields

$$m = |m| e^{i\alpha} \quad \psi = e^{-i\alpha/2} \bar{\psi}$$

and so without loss of generality we can take $m$ to be real and positive.

Equation of motion:
$$0 = \frac{\delta S}{\delta \psi^\dagger} = -i \bar{\sigma}^\mu \partial_\mu \psi + m \psi^\dagger$$

Taking hermitian conjugate:
$$\sigma^{\mu a c} = (I, -\bar{\sigma})$$

are hermitian

$$0 = +i (\sigma^{\mu a c})^* \partial_\mu \psi_c^\dagger + m \psi^a$$

$$\Rightarrow 0 = -i \sigma^{\mu a c} \partial_\mu \psi_c^\dagger + m \psi^a$$

$$\sigma^{\mu a c} = e^{ab} \bar{\sigma}^{\mu b c}$$

We can combine the two equations:
$$0 = -i \sigma^{\mu a c} \partial_\mu \psi_c + m \psi^1$$

$$0 = -i \sigma^{\mu a c} \partial_\mu \psi^1 + m \psi_a$$

which we can write using 4x4 gamma matrices:

$$\left( \begin{array}{cc} m \delta^a_c & \sigma^{\mu a c} \\ -i \sigma^{\mu a c} & m \delta^a_c \end{array} \right) \left( \begin{array}{c} \psi_c \\ \psi^1 \end{array} \right) = 0$$

and defining four-component Majorana field:

$$\Psi \equiv \left( \begin{array}{c} \psi_c \\ \psi^1 \end{array} \right)$$

as:

$$(-i \gamma^\mu \partial_\mu + m) \Psi = 0$$

Dirac equation
using the sigma-matrix relations:
\[
\sigma_{ab}^\mu = (I, \sigma_\mu)
\]
\[
\sigma^{\alpha\mu} = (I, -\sigma_\mu)
\]
\[
(\sigma^\alpha a^\mu + \sigma^\mu a^\alpha)_e^c = -2g^\alpha e_\delta^c \\
(\sigma^\mu a^\nu + \sigma^\nu a^\mu)_e^c = -2g^\mu e_\delta^c
\]
we see that
\[
\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}
\]
and we know that we needed 4 such matrices; recall:
\[
i\hbar \frac{\partial}{\partial t} \psi_i(x) = \left(-i\hbar c(\alpha^i)_{ab} \partial_j + mc^2(\beta)_{ab}\right) \psi_b(x)
\]
\[
\{\alpha^i, \alpha^j\}_{ab} = 2\delta^{ij}\delta_{ab}, \quad \{\alpha^i, \beta\}_{ab} = 0, \quad (\beta^2)_{ab} = \delta_{ab}
\]
\[
\beta = \gamma^0
\]
\[
\alpha^i = \gamma^0 \alpha^i
\]
\[
(-i\gamma^\mu \partial_\mu + m)\psi = 0
\]
consider a theory of two left-handed spinor fields:
\[
\mathcal{L} = i\psi^\dagger_1 \sigma^\mu \partial_\mu \psi_1 - \frac{1}{2} m\psi_1 \psi_1 - \frac{1}{2} m\psi_1^\dagger \psi_1^\dagger
\]
invariant under the SO(2) transformation:
\[
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\cos \alpha & \sin \alpha \\
-\sin \alpha & \cos \alpha
\end{pmatrix}
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}
\]
rewritten in a form that is manifestly U(1) symmetric:
\[
\begin{align*}
\chi &= \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2) \\
\xi &= \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2)
\end{align*}
\]
\[
\mathcal{L} = i\chi^\dagger \sigma^\mu \partial_\mu \chi + i\xi^\dagger \sigma^\mu \partial_\mu \xi - m\chi \xi - m\xi^\dagger \chi^\dagger
\]
Equations of motion for this theory:
\[
\begin{pmatrix}
m\delta_\alpha^c \\
-i\sigma^{\alpha\beta}_c \partial_\mu
\end{pmatrix}
\begin{pmatrix}
\chi_c \\
\xi^{\dagger c}
\end{pmatrix}
= 0
\]
we can define a four-component Dirac field:
\[
\Psi \equiv \begin{pmatrix}
\chi_c \\
\xi^{\dagger c}
\end{pmatrix}
\]
\[
(-i\gamma^\mu \partial_\mu + m)\Psi = 0
\]
we want to write the lagrangian in terms of the Dirac field:
\[
\Psi^\dagger = (\chi^\dagger, \xi^\dagger)
\]
Let's define:
\[
\begin{align*}
\overline{\Psi} &\equiv \Psi^\dagger \beta = (\xi^\alpha, \chi^\dagger) \\
\beta &\equiv \begin{pmatrix}
0 & \delta^c_\delta \\
\delta^c_\alpha & 0
\end{pmatrix}
\end{align*}
\]
numerically:
\[
\beta = \gamma^0
\]
but different spinor index structure
Then we find:
\[
\overline{\Psi} \Psi = \xi^a \chi_a + \chi^a \xi^{\dagger a}
\]
\[
\overline{\Psi} \gamma^\mu \partial_\mu \Psi = \xi^a \sigma^{\mu\alpha}_a \partial_\mu \xi^{\dagger c} + \chi^{\dagger a} \sigma^{\mu\alpha}_a \partial_\mu \chi_a
\]
\[
A\partial B = -(\partial A)B + \partial (AB)
\]
\[
\xi^a \sigma^{\mu\alpha}_a \partial_\mu \xi^{\dagger c} = -\partial_\mu (\xi^a \sigma^{\mu\alpha}_a \xi^{\dagger c}) + \partial_\mu (\xi^a \sigma^{\mu\alpha}_a \xi^{\dagger c})
\]
\[
\partial_\mu (\xi^a \sigma^{\mu\alpha}_a \xi^{\dagger c}) = +\xi^{\dagger c} \sigma^{\mu\alpha}_a \partial_\mu \xi^a = +\xi^{\dagger c} \sigma^{\mu\alpha}_a \partial_\mu \xi^a
\]
Thus we have:
\[
\overline{\Psi} \gamma^\mu \partial_\mu \Psi = \chi^{\dagger} \sigma^\mu \partial_\mu \chi + \xi^{\dagger} \sigma^\mu \partial_\mu \xi + \partial_\mu (\xi \sigma^{\mu\alpha} \xi^{\dagger})
\]
Thus the lagrangian can be written as:

$$\mathcal{L} = i\chi^\dagger \gamma^\mu \partial_\mu \chi + i\xi^\dagger \gamma^\mu \partial_\mu \xi - m\chi\xi - m\xi^\dagger \chi^\dagger$$

The U(1) symmetry is obvious:

$$\Psi \rightarrow e^{-i\alpha} \Psi$$
$$\bar{\Psi} \rightarrow e^{+i\alpha} \bar{\Psi}$$

The Nether current associated with this symmetry is:

$$j^\mu = \bar{\Psi} \gamma^\mu \Psi = \chi^\dagger \sigma^\mu \chi - \xi^\dagger \sigma^\mu \xi$$

later we will see that this is the electromagnetic current

There is an additional discrete symmetry that exchanges the two fields, charge conjugation:

$$C^{-1}\chi_a(x)C = \xi_a(x)$$
$$C^{-1}\xi_a(x)C = \chi_a(x)$$

unitary charge conjugation operator

we want to express it in terms of the Dirac field:

Let's define the charge conjugation matrix:

$$C \equiv \begin{pmatrix} \varepsilon_{ac} & 0 \\ 0 & \varepsilon^{\dot{d}c} \end{pmatrix}$$

then

$$\Psi^C \equiv C\bar{\Psi}^T = \begin{pmatrix} \xi_a \\ \chi^\dagger \end{pmatrix}$$

and we have:

$$C^{-1}\Psi(x)C = \Psi^C(x)$$

The charge conjugation matrix has following properties:

$$C^T = C \dagger = C^{-1} = -C$$

it can also be written as:

$$C = \begin{pmatrix} -\varepsilon_{ac} & 0 \\ 0 & -\varepsilon^{\dot{d}c} \end{pmatrix}$$

and then we find a useful identity:

$$C^{-1}\gamma^\mu C = \begin{pmatrix} e^{ab} & 0 \\ 0 & e_{ab} \end{pmatrix} \begin{pmatrix} \delta_{bc}^\mu & 0 \\ 0 & \delta^{bc}_{\mu} \end{pmatrix} \begin{pmatrix} e_{ce} & 0 \\ 0 & e^{ce} \end{pmatrix}$$

transposed form of

similar to a real scalar field

we get:

$$\Psi \equiv \begin{pmatrix} \psi_c \\ \psi^{\dot{c}} \end{pmatrix}$$

Majorana field is its own conjugate:

$$\Psi^C = \Psi$$

Following the same procedure with:

$$\chi \rightarrow \psi$$
$$\xi \rightarrow \psi$$

we get:

$$\mathcal{L} = \frac{i}{2} \bar{\Psi} \gamma^\mu \partial_\mu \Psi - \frac{1}{2} m\bar{\Psi}\Psi$$

does not incorporate the Majorana condition

incorporating the Majorana condition, we get:

$$\mathcal{L} = \frac{i}{2} \bar{\Psi}^T C \gamma^\mu \partial_\mu \Psi - \frac{1}{2} m\bar{\Psi}^T C \Psi$$

lagrangian for a Majorana field
If we want to go back from 4-component Dirac or Majorana fields to the two-component Weyl fields, it is useful to define a projection matrix:

$$\gamma_5 \equiv \begin{pmatrix} -\delta_a^c & 0 \\ 0 & +\delta_a^c \end{pmatrix}$$

just a name

We can define left and right projection matrices:

$$P_L \equiv \frac{1}{2}(1 - \gamma_5) = \begin{pmatrix} \delta_a^c & 0 \\ 0 & 0 \end{pmatrix}$$

$$P_R \equiv \frac{1}{2}(1 + \gamma_5) = \begin{pmatrix} 0 & 0 \\ 0 & \delta_a^c \end{pmatrix}$$

And for a Dirac field we find:

$$P_L \Psi = \begin{pmatrix} \chi_c \\ 0 \end{pmatrix} \quad \Psi = \begin{pmatrix} \chi_c \\ \xi^\dagger \end{pmatrix}$$

The gamma-5 matrix can be also written as:

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$=-\frac{i}{24}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$$

$$\varepsilon_{0123} = -1$$

Finally, let's take a look at the Lorentz transformation of a Dirac or Majorana field:

$$U(\Lambda)^{-1}\psi_\alpha(x)U(\Lambda) = \Lambda(\alpha)^{\dot{c}}_\gamma \psi_\gamma(\Lambda^{-1}x)$$

$$U(\Lambda)^{-1}\psi_\dot{a}(x)U(\Lambda) = R(\alpha)^{\dot{c}}_\dot{a} \psi_\dot{c}(\Lambda^{-1}x)$$

$$D(1+\delta\omega) = 1 + \frac{i}{2}\delta\omega_{\mu\nu}S^{\mu\nu}$$

$$\frac{i}{4}[\gamma^\mu, \gamma^\nu] = \begin{pmatrix} (S_L^{\mu\nu})_{\alpha}^c & 0 \\ 0 & -(S_R^{\mu\nu})_{\dot{c}}^{\dot{a}} \end{pmatrix} \equiv S^{\mu\nu}$$

$$\gamma^\nu = \begin{pmatrix} 0 & \sigma^\nu_{\dot{a}c} \\ \sigma^{\nu\dot{a}}_{\alpha c} & 0 \end{pmatrix}$$

compensates for $\dot{c} = -\dot{c}$
For a four-component Dirac field we found:
\[
\mathcal{L} = i\chi^\dagger \sigma^\mu \partial_\mu \chi + i\xi^\dagger \sigma^\mu \partial_\mu \xi - m(\chi \xi + \xi^\dagger \chi^\dagger)
\]
\[
= i\bar{\Psi} \gamma^\mu \partial_\mu \Psi - m\bar{\Psi}\Psi .
\]
\[
\Psi \equiv \begin{pmatrix} \chi_c \\ \xi_c \end{pmatrix},
\]
\[
\bar{\Psi} \equiv \Psi^\dagger \beta = (\xi^a, \chi^a),
\]
and the corresponding canonical anticommutation relations are:
\[
\{\psi_a(x, t), \psi^\dagger_b(y, t)\} = \delta^{ab} \delta^3(x - y),
\]
\[
\{\psi_a(x, t), \psi_b(y, t)\} = (\gamma^0)_{\alpha\beta} \delta^3(x - y)
\]
\[
\gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix}
\]
can be also derived directly from \(\partial \mathcal{L}/\partial (\partial_\mu \Psi) = i\bar{\Psi} \gamma^0, ...\)

Now we want to find solutions to the Dirac equation:
\[
(-i\partial_\mu + m)\Psi = 0
\]
where we used the Feynman slash:
\[
\not{\partial} = a_\mu \gamma^\mu
\]
\[
\not{\partial} \not{\partial} = a_\mu a^\mu \gamma^\mu \gamma^\nu + \frac{1}{2} [\gamma^\mu, \gamma^\nu]
\]
\[
= a_\mu a^\mu (-g^{\mu\nu} + \frac{1}{2} [\gamma^\mu, \gamma^\nu])
\]
\[
= -a^2 .
\]
then we find:
\[
0 = (i\partial + m)(-i\partial + m)\Psi
\]
\[
= (\not{\partial} + m^2)\Psi
\]
\[
= (-\partial^2 + m^2)\Psi .
\]
the Dirac (or Majorana) field satisfies the Klein-Gordon equation and so the Dirac equation has plane-wave solutions!

For a four-component Majorana field we found:
\[
\mathcal{L} = i\psi^\dagger \sigma^\mu \partial_\mu \psi - \frac{1}{2} m(\psi \psi + \psi^\dagger \psi^\dagger)
\]
\[
= i\frac{1}{2} \bar{\psi} \gamma^\mu \partial_\mu \psi - \frac{1}{2} m\bar{\psi}\psi
\]
\[
= i\frac{1}{2} \bar{\psi}^\dagger C\gamma^\mu \partial_\mu \psi - \frac{1}{2} m\bar{\psi}^\dagger C\psi .
\]
\[
\bar{\psi} \equiv \psi^\dagger \beta = (\psi^a, \psi^a),
\]
\[
\bar{\psi} = \bar{\psi}^\dagger C
\]
\[
c \equiv \begin{pmatrix} -\epsilon^{\alpha\beta} & 0 \\ 0 & \epsilon^{\alpha\beta} \end{pmatrix}
\]
and the corresponding canonical anticommutation relations are:
\[
\{\psi_\alpha(x, t), \psi_\beta(y, t)\} = (C \gamma^0)_{\alpha\beta} \delta^3(x - y),
\]
\[
\{\psi_\alpha(x, t), \bar{\psi}_\beta(y, t)\} = (\gamma^0)_{\alpha\beta} \delta^3(x - y),
\]
Consider a solution of the form:
\[
\Psi(x) = u(p)e^{ipx} + v(p)e^{-ipx}
\]
\[
p^0 = \omega \equiv (p^2 + m^2)^{1/2}
\]
\[
(-i\partial + m)\Psi = 0
\]
plugging it into the Dirac equation gives:
\[
(\not{\partial} + m)u(p)e^{ipx} + (-\not{\partial} + m)v(p)e^{-ipx} = 0
\]
that requires:
\[
(\not{\partial} + m)u(p) = 0
\]
\[
(\not{\partial} - m)v(p) = 0
\]
each eq. has two solutions (later)
The general solution of the Dirac equation is:
\[
\Psi(x) = \sum_{s=\pm} \int dp \left[ b_s(p)u_s(p)e^{ipx} + d_s(p)v_s(p)e^{-ipx} \right]
\]
\[
dp \equiv \frac{dp}{(2\pi)^{3/2}}
\]
Spinor technology

The four-component spinors obey equations:

\[(\not{\psi} + m)u_s(\mathbf{p}) = 0\]
\[(-\not{\psi} + m)v_s(\mathbf{p}) = 0\]

In the rest frame, \(\mathbf{p} = 0\) we can choose:

\[u_+(0) = \sqrt{m} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad u_-(0) = \sqrt{m} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix},\]
\[v_+(0) = \sqrt{m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad v_-(0) = \sqrt{m} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.\]

convenient normalization and phase

let us also compute the barred spinors:

\[\bar{u}_s(\mathbf{p}) \equiv u_s^\dagger(\mathbf{p})\beta, \quad \beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\]
\[\bar{v}_s(\mathbf{p}) \equiv v_s^\dagger(\mathbf{p})\beta\]

we get:

\[\bar{u}_+(0) = \sqrt{m} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \bar{u}_-(0) = \sqrt{m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \bar{v}_+(0) = \sqrt{m} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \quad \bar{v}_-(0) = \sqrt{m} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}.\]

We can find spinors at arbitrary 3-momentum by applying the matrix that corresponds to the boost:

\[U(\Lambda)^{-1}\psi(\mathbf{x})U(\Lambda) = D(\Lambda)\psi(\Lambda^{-1}\mathbf{x})\]

\[D(\Lambda) = \exp(i\eta \hat{\mathbf{p}} \cdot \mathbf{K})\]

homework

\[K^j = \frac{i}{4}[\gamma^j, \gamma^0] = \frac{i}{4}\gamma^j\gamma^0\]
\[S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]\]
\[\eta \equiv \sinh^{-1}(p/m)\]

we find:

\[u_s(\mathbf{p}) = \exp(i\eta \hat{\mathbf{p}} \cdot \mathbf{K})u_s(0)\]
\[v_s(\mathbf{p}) = \exp(i\eta \hat{\mathbf{p}} \cdot \mathbf{K})v_s(0)\]

and similarly:

\[\bar{u}_s(\mathbf{p}) = \bar{u}_s(0)\exp(-i\eta \hat{\mathbf{p}} \cdot \mathbf{K})\]
\[\bar{v}_s(\mathbf{p}) = \bar{v}_s(0)\exp(-i\eta \hat{\mathbf{p}} \cdot \mathbf{K})\]

\[\bar{K}^j = K^j\]
\[\bar{A} = \beta A^\dagger \beta\]
For any combination of gamma matrices we define:

\[ \overline{A} \equiv \beta A^\dagger \beta \]

It is straightforward to show:

\[ \overline{\gamma^\mu} = \gamma^\mu, \]
\[ S^{\mu\nu} = S^{\nu\mu}, \]
\[ \overline{i \gamma_5} = i \gamma_5, \]
\[ \gamma^\mu \gamma_5 = \gamma^\mu \gamma_5, \]
\[ \overline{i \gamma_5 S^{\mu\nu}} = i \gamma_5 S^{\mu\nu}. \]

Useful identities (Gordon identities):

\[ 2m \overline{u}_{s'}(p') \gamma^\mu u_s(p) = \overline{u}_{s'}(p') \left[ (p' + p)^\mu - 2i S^{\mu\nu} (p' - p)_\nu \right] u_s(p) \]
\[ -2m \overline{v}_{s'}(p') \gamma^\mu v_s(p) = \overline{v}_{s'}(p') \left[ (p' + p)^\mu - 2i S^{\mu\nu} (p' - p)_\nu \right] v_s(p) \]

Proof:

\[ \gamma^\mu \gamma_5 = \frac{1}{2} \{ \gamma^\mu, \gamma_5 \} + \frac{1}{2} [\gamma^\mu, \gamma_5] = -p^\mu - 2i S^{\mu\nu} p_\nu \]
\[ \gamma_5^\mu \gamma_5 = \frac{1}{2} \{ \gamma_5^\mu, \gamma_5 \} - \frac{1}{2} [\gamma_5^\mu, \gamma_5] = -p^\mu + 2i S^{\mu\nu} p_\nu \]

add the two equations, and sandwich them between spinors, \( \{ \gamma^\mu, \gamma_5 \} = -2g^{\mu\nu} \)
and use:

\[ (\gamma + m)u(p) = 0 \]
\[ (-\gamma + m)v(p) = 0 \]
\[ \overline{u}_s(p)(\gamma + m) = 0 \]
\[ \overline{v}_s(p)(-\gamma + m) = 0 \]

An important special case \( p' = p \):

\[ \overline{u}_{s'}(p') \gamma^\mu u_s(p) = 2p^\mu \delta_{s's} \]
\[ \overline{v}_{s'}(p') \gamma^\mu v_s(p) = 2p^\mu \delta_{s's} \]

One can also show:

\[ \overline{u}_{s'}(p') \gamma^0 v_s(-p) = 0 \]
\[ \overline{v}_{s'}(p') \gamma^0 u_s(-p) = 0 \]

Gordon identities with gamma-5:

\[ \overline{u}_{s'}(p') \left[ (p' + p)^\mu - 2i S^{\mu\nu} (p' - p)_\nu \right] \gamma_5 u_s(p) = 0 \]
\[ \overline{v}_{s'}(p') \left[ (p' + p)^\mu - 2i S^{\mu\nu} (p' - p)_\nu \right] \gamma_5 v_s(p) = 0 \]
We will find very useful the spin sums of the form:
\[ \sum_{s=\pm} u_s(p) \bar{u}_s(p) \]

which can be directly calculated but we will find the correct form by the following argument: the sum over spin removes all the memory of the spin-quantization axis, and the result can depend only on the momentum four-vector and gamma matrices with all indices contracted.

In the rest frame, \( \gamma' = -m \gamma^0 \), we have:
\[ \sum_{s=\pm} u_s(0) \bar{u}_s(0) = m \gamma^0 + m \]
\[ \sum_{s=\pm} v_s(0) \bar{v}_s(0) = m \gamma^0 - m \]

Thus we conclude:
\[ \sum_{s=\pm} u_s(p) \bar{u}_s(p) = -\gamma' + m \]
\[ \sum_{s=\pm} v_s(p) \bar{v}_s(p) = -\gamma' - m \]

if instead of the spin sum we need just a specific spin product, e.g.,
\[ u_+(p) \bar{u}_+(p) \]

we can get it using appropriate spin projection matrices:

in the rest frame we have
\[ \frac{1}{2}(1 + 2sS_z)u_s(0) = \delta_{ss'}u_s(0) \]
\[ \frac{1}{2}(1 - 2sS_z)u_s(0) = \delta_{ss'}v_s(0) \]

the spin matrix \( S_z = \frac{1}{2}\gamma^1\gamma^2 \) can be written as:
\[ S_z = -\frac{1}{2} \gamma_5 \gamma^3 \gamma^0 \]
\[ \gamma_5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \]

in the rest frame we can write \( \gamma^0 = -\gamma'/m \) and \( \gamma^3 = \gamma' \) and so we have:
\[ S_z = \frac{1}{2m} \gamma_5 \gamma' \gamma' \]
\[ \sum_{s=\pm} u_s(p) \bar{u}_s(p) = \frac{1}{2} (1 - s \gamma_5 \gamma') (-\gamma' + m) \]
\[ \sum_{s=\pm} v_s(p) \bar{v}_s(p) = \frac{1}{2} (1 - s \gamma_5 \gamma') (-\gamma' - m) \]

Boosting to a different frame we get:
\[ u_s(p) = \exp(\gamma' \cdot \mathbf{K}) u_s(0) \]
\[ v_s(p) = \exp(\gamma' \cdot \mathbf{K}) v_s(0) \]
\[ S_z = \frac{1}{2m} \gamma_5 \gamma' \gamma' \]

Let's look at the situation with 3-momentum in the z-direction:

The component of the spin in the direction of the 3-momentum is called the helicity (a fermion with helicity +1/2 is called right-handed, a fermion with helicity -1/2 is called left-handed).

\[ \frac{1}{m} p^\mu = (\cosh \eta, 0, 0, \sinh \eta) \]
\[ z^\mu = (\sinh \eta, 0, 0, \cosh \eta) \]
\[ z^2 = 1 \]
\[ z \cdot p = 0 \]

In the limit of large rapidity
\[ z^\mu = \frac{1}{m} p^\mu + O(e^{-\eta}) \]
In the limit of large rapidity
\[ z^\mu = \frac{1}{m} p^\mu + O(e^{-\eta}) \]

\[ u_s(p)\overline{u}_s(p) = \frac{1}{2} (1 - s \gamma_5)(-\not{p} + m) \]
\[ v_s(p)\overline{v}_s(p) = \frac{1}{2} (1 - s \gamma_5)(-\not{p} - m) \]

In the extreme relativistic limit the right-handed fermion (helicity +1/2) (described by spinors \( u^+ \) for b-type particle and \( v^- \) for d-type particle) is projected onto the lower two components only (part of the Dirac field that corresponds to the right-handed Weyl field). Similarly left-handed fermions are projected onto upper two components (right-handed Weyl field).

Formulas relevant for massless particles can be obtained from considering the extreme relativistic limit of a massive particle; in particular the following formulas are valid when setting \( m = 0 \):

\[ (\not{p} + m)u_s(p) = 0 \]
\[ (-\not{p} + m)v_s(p) = 0 \]
\[ \overline{u}_s(p)(\not{p} + m) = 0 \]
\[ \overline{v}_s(p)(-\not{p} + m) = 0 \]
\[ \overline{u}_{s'}(p)u_s(p) = +2m \delta_{s's} \]
\[ \overline{v}_{s'}(p)v_s(p) = -2m \delta_{s's} \]
\[ \overline{u}_{s'}(p)v_s(p) = 0 \]
\[ \overline{v}_{s'}(p)u_s(p) = 0 \]

\[ u_s(p)\overline{u}_s(p) \rightarrow \frac{1}{2} (1 + s \gamma_5)(-\not{p}) \]
\[ v_s(p)\overline{v}_s(p) \rightarrow \frac{1}{2} (1 - s \gamma_5)(-\not{p}) \]

Lagrangian for a Dirac field:
\[ \mathcal{L} = i \overline{\Psi} \gamma^\mu \partial_\mu \Psi - m \overline{\Psi} \Psi \]

canonical anticommutation relations:
\[ \{ \Psi_\alpha(x, t), \overline{\Psi}_\beta(y, t) \} = 0 \]
\[ \{ \Psi_\alpha(x, t), \overline{\Psi}_\beta(y, t) \} = (\gamma^0)_{\alpha\beta} \delta^3(x - y) \]

The general solution to the Dirac equation:
\[ \Psi(x) = \sum_{s = \pm} \int \frac{dp}{2\pi} \left[ b_s(p)u_s(p)e^{ipx} + d^*_s(p)\overline{v}_s(p)e^{-ipx} \right] \]

We want to find formulas for creation and annihilation operator:
\[ \Psi(x) = \sum_{s = \pm} \int \frac{dp}{2\pi} \left[ b'_s(p)u_s(p)e^{ipx} + d^*_s(p)\overline{v}_s(p)e^{-ipx} \right] \]

multiply by \( \overline{u}_s(p)\gamma^0 \) on the left:
\[ \overline{u}_s(p)\gamma^0 \Psi(x) = \sum_{s = \pm} \int \frac{dp}{2\pi} \left[ \frac{1}{2\omega} b_{s'}(p)u_{s'}(p) + \frac{1}{2\omega} e^{2i\omega t} d^*_s(p)\overline{v}_{s'}(-p) \right] \]

for the hermitian conjugate we get:
\[ \left[ \overline{u}_s(p)\gamma^0 \Psi(x) \right]^\dagger = \overline{\Psi}(x)\gamma^0 u_s(p) \]

b's are time independent!
\[
\Psi(x) = \sum_{s=\pm} \int dp \left[ b_s(p) u_s(p) e^{ipx} + d^{\dagger}_s(p) \overline{u}_s(p) e^{-ipx} \right]
\]

Similarly for \( d \):

\[
\int d^3x \ e^{ipx} \Psi(x) = \sum_{s=\pm} \left[ \frac{1}{2m} e^{-2i\omega t} b_s(-p) u_s(-p) + \frac{i}{2\omega} d^{\dagger}_s(p) \overline{u}_s(p) \right]
\]

Multiply by \( \overline{u}_s(p) \gamma^0 \) on the left:

\[
\overline{u}_s(p) \gamma^0 u_s(-p) = 2p^\mu \delta_{\nu s}
\]

for the hermitian conjugate we get:

\[
d_s(p) = \int d^3x \ e^{ipx} \overline{\Psi}(x) \gamma^0 u_s(p)
\]

\[
b_s(p) = \int d^3x \ e^{-ipx} \overline{u}_s(p) \gamma^0 \Psi(x)
\]

we can easily work out the anticommutation relations for \( b \) and \( d \) operators:

\[
\{ \Psi_\alpha(x, t), \Psi_\beta(y, t) \} = 0,
\]

\[
\{ \Psi_\alpha(x, t), \overline{\Psi}_\beta(y, t) \} = (\gamma^0)_{\alpha\beta} \delta^0(x - y)
\]

\[
\{ b_s(p), b^{\dagger}_{s'}(p') \} = 0
\]

\[
\{ d_s(p), d^{\dagger}_{s'}(p') \} = 0
\]

\[
\{ b_s(p), d^{\dagger}_{s'}(p') \} = 0
\]

\[
\{ b^{\dagger}_s(p), b_{s'}(p') \} = 0
\]

\[
\{ d^{\dagger}_s(p), d_{s'}(p') \} = 0
\]

\[
\{ b^{\dagger}_s(p), d_{s'}(p') \} = 0
\]

\[
\{ d^{\dagger}_s(p), b_{s'}(p') \} = 0
\]
\[ b_s(p) = \int d^3x \, e^{-ipx} \bar{u}_s(p) \gamma^0 \Psi(x) \quad d_s(p) = \int d^3x \, e^{-ipx} \bar{\Psi}(x) \gamma^0 v_s(p) \]

and finally:

\[ \{\Psi_{\alpha}(x, t), \bar{\Psi}_{\beta}(y, t)\} = 0, \quad \{\Psi_{\alpha}(x, t), \bar{\Psi}_{\beta}(y, t)\} = (\gamma^0)_{\alpha\beta} \delta^3(x - y) \]

\[ \{b_s(p), d_{s'}(p')\} = \int d^3x \, d^3y \, e^{-ipx - ip'y} \bar{u}_s(p) \gamma^0 \{\Psi(x), \bar{\Psi}(y)\} \gamma^0 v_{s'}(p') \]

\[ = \int d^3x \, e^{-i(p + p')x} \bar{u}_s(p) \gamma^0 \gamma^0 v_{s'}(p') \]

\[ = (2\pi)^3 \delta^3(p + p') \bar{u}_s(p) \gamma^0 v_{s'}(-p) \]

\[ = 0. \quad \bar{u}_{s'}(p) \gamma^0 v_s(-p) = 0 \]

We want to calculate the hamiltonian in terms of the \(b\) and \(d\) operators; in the four-component notation we would find:

\[ H = \int d^3x \, \bar{\Psi}(-i\gamma^j \partial_j + m) \Psi \]

let's start with:

\[ (-i\gamma^j \partial_j + m) \Psi = \sum_{s = \pm} \int d\tilde{p} \, \bar{b}_s(p) u_s(p) e^{ipx} + d_s(p) v_s(p) e^{-ipx} \]

\[ = \sum_{s = \pm} \int d\tilde{p} \, \left[ b_s(p)(+\gamma^j \partial_j + m) u_s(p) e^{ipx} + d_s(p)(-\gamma^j \partial_j + m) v_s(p) e^{-ipx} \right] \]

\[ (\hat{p} + m) u_s(p) = 0 \]

\[ (-\hat{p} + m) v_s(p) = 0 \]

\[ \Psi(x) = \sum_{s = \pm} \int d\tilde{p} \, [b_s(p) u_s(p) e^{ipx} + d_s(p) v_s(p) e^{-ipx}] \]

\[ H = \int d^3x \, \bar{\Psi}(-i\gamma^j \partial_j + m) \Psi \]

\[ = \sum_{s, s'} \int d\tilde{p} \, d\tilde{p}' \, d^3x \, \omega \left[ b_{s'}(p') \bar{u}_{s'}(p') \gamma^0 u_s(p) e^{-i(p' - p)x} \right. \]

\[ = \sum_{s, s'} \int d\tilde{p} \, d\tilde{p}' \, d^3x \, \omega \left[ b_{s'}(p') \bar{u}_{s'}(p') \gamma^0 u_s(p) e^{-i(p' - p)x} \right. \]

\[ - b_s(p) d_s(p) \bar{u}_s(p) \gamma^0 v_s(p) e^{-i(p + p')x} \]

\[ + d_{s'}(p') b_s(p) \bar{v}_{s'}(p') \gamma^0 u_s(p) e^{i(p' + p)x} \]

\[ - d_{s'}(p') d_{s'}(p) \bar{v}_{s'}(p') \gamma^0 v_s(p) e^{i(p' - p)x} \]

\[ = \sum_{s} \int d\tilde{p} \, \left[ b_{s'}(p) b_s(p) - d_{s'}(p) d_s(p) \right]. \]
finally, we find:

\[ H = \sum_{s=\pm} \int \frac{d^3p}{(2\pi)^3} \frac{\omega}{2} \left[ b_s^\dagger(p)b_s(p) + d_s^\dagger(p)d_s(p) \right] - 4E_0V \]

\[
V = (2\pi)^3 \delta^3(0) = \int d^3x
\]

\[ E_0 = \frac{1}{2} (2\pi)^{-3} \int d^3k \omega \]

four times the zero-point energy of a scalar field and opposite sign!

we will assume that the zero-point energy is cancelled by a constant term

\[
H = \sum_{s=\pm} \int \frac{d^3p}{(2\pi)^3} \frac{\omega}{2} \left[ b_s^\dagger(p)b_s(p) + d_s^\dagger(p)d_s(p) \right] - 4E_0V
\]

b- and d-type particles are distinguished by the value of the charge:

\[
Q = \int d^3x \, j^0
\]

\[ j^\mu = \bar{\Psi} \gamma^\mu \Psi \]

very similar calculation as for the hamiltonian; we get:

\[
Q = \int d^3x \, \bar{\Psi} \gamma^0 \Psi = \sum_{s=\pm} \int \frac{d^3p}{(2\pi)^3} \left[ b_s^\dagger(p)b_s(p) + d_s(p)d_s^\dagger(p) \right]
\]

\[
E_0 = \frac{1}{2} (2\pi)^{-3} \int d^3k \omega
\]

counts the number of b-type particles - the number of d-type particles

(later, the electron will be a b-type particle and the positron a d-type particle)

\[
\Psi(x) = \sum_{s=\pm} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\sqrt{\omega}} \left[ b_s(p)u_s(p)e^{ipx} + d_s^\dagger(p)v_s(p)e^{-ipx} \right]
\]

spin-1/2 states:

vacuum:

\[ |0\rangle \]

\[ b_s(p)|0\rangle = d_s(p)|0\rangle = 0 \]

b-type particle with momentum \( p \), energy \( \omega = (p^2 + m^2)^{1/2} \), and spin \( S_s = \frac{1}{2} s \):

\[ |p, s, +\rangle = b_s^\dagger(p)|0\rangle \]

labels the charge of a particle

d-type particle with momentum \( p \), energy \( \omega = (p^2 + m^2)^{1/2} \), and spin \( S_s = \frac{1}{2} s \):

\[ |p, s, -\rangle = d_s^\dagger(p)|0\rangle \]
we have just used:

\[ C \tilde{u}_s(p)^T = v_s(p) \]
\[ C \tilde{v}_s(p)^T = u_s(p) \]

Proof:

\[
C = \begin{pmatrix}
0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
+1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0
\end{pmatrix}
\]

\[
\tilde{u}_s(0) = \sqrt{m} \begin{pmatrix} 1, 0, 1, 0 \end{pmatrix} ,
\tilde{v}_s(0) = \sqrt{m} \begin{pmatrix} 0, 1, 0, 1 \end{pmatrix} ,
\tilde{u}_s(0) = \sqrt{m} \begin{pmatrix} 0, -1, 0, 1 \end{pmatrix} ,
\tilde{v}_s(0) = \sqrt{m} \begin{pmatrix} 0, 1, -1, 0 \end{pmatrix} .
\]

by direct calculation:

\[
C \tilde{u}_s(0)^T = v_s(0) \]
\[
C \tilde{v}_s(0)^T = u_s(0) \]

boosting to any frame we get:

\[
\tilde{u}_s(p) = \tilde{u}_s(0) \exp(-i\eta \cdot p \cdot K)
\]
\[
\tilde{v}_s(p) = \tilde{v}_s(0) \exp(-i\eta \cdot p \cdot K)
\]

\[
u_s(p) = \exp(i\eta \cdot p \cdot K) u_s(0)
\]
\[
u_s(p) = \exp(i\eta \cdot p \cdot K) v_s(0)
\]

\[
C \tilde{u}_s(p)^T = v_s(p)
\]
\[
C \tilde{v}_s(p)^T = u_s(p)
\]

\[
\beta C = -\beta\gamma^3 C = \left(\gamma^\nu\right)^T
\]
\[
C^{-1} \gamma^\nu C = \left(\gamma^\nu\right)^T
\]
\[
K^j = \frac{i}{4}[\gamma^j, \gamma^0] = \frac{i}{2} \gamma^j \gamma^0
\]
\[
C^{-1} K^j C = \left(\gamma^j\right)^T
\]

The hamiltonian for the Majorana field is:

\[
H = \frac{1}{2} \int d^3x \left[ \Psi^T C (-i \gamma^\nu \partial_\nu + m) \Psi\right]
\]
\[
= \frac{1}{2} \int d^3x \left[ \tilde{\Psi}(i \gamma^\nu \partial_\nu + m) \tilde{\Psi} \right]
\]

and repeating the same manipulations as for the Dirac field we would find:

\[
H = \frac{1}{2} \sum_{s=\pm} \int d^3p \omega \left[ b_s^\dagger(p) b_s(p) - b_s(p) b_s^\dagger(p) \right] - 2 \mathcal{E}_0 V.
\]

\[
\{b_s(p), b_{s'}(p')\} = (2\pi)^3 \delta^3(p - p') 2 \omega \delta_{ss'}
\]

We have found that a Majorana field can be written:

\[
\Psi(x) = \sum_{s=\pm} \int \tilde{d}p \left[ b_s(p) u_s(p) e^{ipx} + b_s^\dagger(p) v_s(p) e^{-ipx} \right]
\]

canonical anticommutation relations:

\[
\{\Psi_\alpha(x, t), \Psi_\beta(y, t)\} = (C\gamma^0)_{\alpha\beta} \delta^3(x - y)
\]
\[
\{\Psi_\alpha(x, t), \overline{\Psi}_\beta(y, t)\} = (\gamma^0)_{\alpha\beta} \delta^3(x - y)
\]

translate into:

\[
\{b_s(p), b_{s'}(p')\} = 0
\]
\[
\{b_s(p), b_{s'}^\dagger(p')\} = (2\pi)^3 \delta^3(p - p') 2 \omega \delta_{ss'}
\]

V = (2\pi)^3 \delta^3(0) = \int d^3x
\]
\[
\mathcal{E}_0 = \frac{1}{2} (2\pi)^{-3} \int d^3k \omega
\]

we will assume that the zero-point energy is cancelled by a constant term.

two times the zero-point energy of a scalar field and opposite sign!