Two particle elastic scattering at 1-loop

Let’s use our rules to calculate two-particle elastic scattering amplitude in $\varphi^3$ theory in 6 dimensions including all one-loop corrections:

For the amplitude at tree level we have found:

$$iT_{\text{tree}} = \frac{1}{i} (ig)^2 \left[ \Delta(-s) + \Delta(-t) + \Delta(-u) \right]$$

$s \equiv -(k_1+k_2)^2 = -(k'_1+k'_2)^2$

$t \equiv -(k_1-k'_1)^2 = -(k_2-k'_2)^2$

$u \equiv -(k_1-k'_2)^2 = -(k_2-k'_1)^2$

$s + t + u = m_1^2 + m_2^2 + m_1'^2 + m_2'^2$

$\Delta(-s) = \frac{1}{(-s + m^2 - i\epsilon)}$

positive

negative
The exact scattering amplitude is given by the diagrams:

\[ i\mathcal{T}_{1\text{-}\text{loop}} = \frac{1}{i} \left( [iV_3(s)]^2 \tilde{\Delta}(-s) + [iV_3(t)]^2 \tilde{\Delta}(-t) + [iV_3(u)]^2 \tilde{\Delta}(-u) \right) + iV_4(s, t, u), \]

exact 4-point vertex

exact 3-point vertex

exact propagator

external lines contribute only the residue of the pole at \( k^2 = -m^2 \); which is one
At one loop level we have:

\[ i\mathcal{T}_{1\text{-loop}} = \frac{1}{i} \left( [i \mathbf{V}_3(s)]^2 \tilde{\Delta}(-s) + [i \mathbf{V}_3(t)]^2 \tilde{\Delta}(-t) + [i \mathbf{V}_3(u)]^2 \tilde{\Delta}(-u) \right) + i \mathbf{V}_4(s, t, u), \]

\[ \tilde{\Delta}(-s) = \frac{1}{-s + m^2 - \Pi(-s)}, \]

\[ \Pi(-s) = \frac{1}{2} \alpha \int_0^1 dx D_2(s) \ln \left( \frac{D_2(s)}{D_0} \right) - \frac{1}{12} \alpha (-s + m^2), \]

\[ \mathbf{V}_3(s)/g = 1 - \frac{1}{2} \alpha \int dF_3 \ln \left( \frac{D_3(s)}{m^2} \right), \]

\[ \mathbf{V}_4(s, t, u) = \frac{1}{6} g^2 \alpha \int dF_4 \left[ \frac{1}{D_4(s, t)} + \frac{1}{D_4(t, u)} + \frac{1}{D_4(u, s)} \right]. \]

\[ \mathbf{V}_3(k_1, k_2, k_3)/g = 1 - \frac{1}{2} \alpha \int dF_3 \ln(D/m^2) - \kappa_C \alpha + O(\alpha^2) \]

\[ D = x_1(1-x_1)k_1^2 + x_2(1-x_2)k_2^2 + 2x_1x_2k_1\cdot k_2 + m^2 \]

\[ = x_3x_1k_1^2 + x_3x_2k_2^2 + x_1x_2k_3^2 + m^2, \]

\[ \text{two external momenta are on shell} \]

\[ D_2(s) = -x(1-x)s + m^2, \]

\[ D_0 = +[1-x(1-x)]m^2, \]

\[ D_3(s) = -x_1x_2s + [1-(x_1+x_2)x_3]m^2, \]

\[ D_4(s, t) = -x_1x_2s - x_3x_4t + [1-(x_1+x_2)(x_3+x_4)]m^2. \]
We can get some intuition by looking at high-energy limit (neglecting mass terms where possible):

**Propagator:**

\[ 
\tilde{\Delta}(-s) = \frac{1}{-s + m^2 - \Pi(-s)}, 
\]

\[ 
\Pi(-s) = \frac{1}{2} \alpha \int_0^1 dx \, D_2(s) \ln \left( \frac{D_2(s)}{D_0} \right) - \frac{1}{12} \alpha (-s + m^2) 
\]

\[ 
D_2(s) = -x(1-x)s + m^2 
\]

\[ 
D_0 = +[1-x(1-x)]m^2 
\]

\[ 
\Pi(-s) = -\frac{1}{2} \alpha s \int_0^1 dx \, x(1-x) \left[ \ln \left( \frac{-s}{m^2} \right) + \ln \left( \frac{x(1-x)}{1-x(1-x)} \right) \right] + \frac{1}{12} \alpha s 
\]

\[ 
= -\frac{1}{12} \alpha s \left[ \ln(-s/m^2) + 3 - \pi \sqrt{3} \right]. 
\]

\[ 
\tilde{\Delta}(-s) = \frac{1}{-s - \Pi(-s)} 
\]

\[ 
= -\frac{1}{s} \left( 1 + \frac{1}{12} \alpha \left[ \ln(-s/m^2) + 3 - \pi \sqrt{3} \right] \right) + O(\alpha^2) 
\]

- \( s \) is positive
- Correct branch obtained by \( s \to s + i\epsilon \)

\[ 
\ln(-s) = \ln s - i\pi 
\]
Three-point vertex:

\[ V_3(s)/g = 1 - \frac{1}{2} \alpha \int dF_3 \ln \left( \frac{D_3(s)}{m^2} \right) \]

\[ D_3(s) = -x_1x_2s + [1-(x_1+x_2)x_3]m^2 \]

we get:

\[ V_3(s)/g = 1 - \frac{1}{2} \alpha \int dF_3 \left[ \ln(-s/m^2) + \ln(x_1x_2) \right], \]

\[ = 1 - \frac{1}{2} \alpha \left[ \ln(-s/m^2) - 3 \right], \]

Four-point vertex:

\[ V_4(s,t,u) = \frac{1}{6} g^2 \alpha \int dF_4 \left[ \frac{1}{D_4(s,t)} + \frac{1}{D_4(t,u)} + \frac{1}{D_4(u,s)} \right] \]

\[ D_4(s,t) = -x_1x_2s - x_3x_4t + [1-(x_1+x_2)(x_3+x_4)]m^2 \]

\[ \int \frac{dF_4}{D_4(s,t)} = - \frac{3}{s+t} \left( \pi^2 + \left[ \ln(s/t) \right]^2 \right) \]

\[ = + \frac{3}{u} \left( \pi^2 + \left[ \ln(s/t) \right]^2 \right), \]
Note:

when evaluating integrals over Feynman variables we use:

$$\int dF_n f(x) = (n-1)! \int_0^1 dx_1 \ldots dx_n \delta(x_1+\ldots+x_{n-1}-1) f(x)$$

$$= (n-1)! \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \ldots \int_0^{1-x_1-\ldots-x_{n-2}} dx_{n-1}$$

$$\times f(x) \Bigg|_{x_n=1-x_1-\ldots-x_{n-1}}.$$
some useful integrals:

\[ \int \frac{1}{(x + a)(x + b)} \, dx = \frac{1}{b - a} \ln \frac{a + x}{b + x}, \quad a \neq b \]

\[ \int \ln(ax + b) \, dx = \left( x + \frac{b}{a} \right) \ln(ax + b) - x + C, \quad a \neq 0 \]

\[ \int x \ln(ax + b) \, dx = \frac{bx}{2a} - \frac{1}{4} x^2 + \frac{1}{2} \left( x^2 - \frac{b^2}{a^2} \right) \ln(ax + b) + C \]
Putting it all together:

\[ \tilde{\Delta}(-s) = -\frac{1}{s} \left( 1 + \frac{1}{12} \alpha \left[ \ln(-s/m^2) + 3 - \pi \sqrt{3} \right] \right) + O(\alpha^2) \]

\[ V_3(s)/g = 1 - \frac{1}{2} \alpha \left[ \ln(-s/m^2) - 3 \right] \]

\[ V_4(s,t,u) = \frac{1}{6} g^2 \alpha \int \frac{dF_4}{D_4(s,t)} + \frac{1}{D_4(t,u)} + \frac{1}{D_4(u,s)} \]

\[ = + \frac{3}{u} \left( \pi^2 + \left[ \ln(s/t) \right]^2 \right) \]

\[ i\mathcal{T}_{1\text{-loop}} = \frac{1}{i} \left( [iV_3(s)]^2 \tilde{\Delta}(-s) + [iV_3(t)]^2 \tilde{\Delta}(-t) + [iV_3(u)]^2 \tilde{\Delta}(-u) \right) \]

\[ + iV_4(s,t,u), \]

we get:

\[ \mathcal{T}_{1\text{-loop}} = g^2 \left[ F(s,t,u) + F(t,u,s) + F(u,s,t) \right] \]

\[ F(s,t,u) \equiv -\frac{1}{s} \left( 1 - \frac{11}{12} \alpha \left[ \ln(-s/m^2) + c \right] - \frac{1}{2} \alpha \left[ \ln(t/u) \right]^2 \right) \]

\[ c = (6\pi^2 + \pi \sqrt{3} - 39)/11 = 2.33 \]

**typical result:**

tree level amplitude is corrected by powers of logs of kinematic variables!
Infrared divergences

We calculated two-particle elastic scattering amplitude in $\varphi^3$ theory in 6 dimensions including all one-loop corrections:

$$i T_{\text{tree}} = \frac{1}{i} (ig)^2 \left[ \tilde{\Delta}(-s) + \tilde{\Delta}(-t) + \tilde{\Delta}(-u) \right]$$

$$T_{1\text{-loop}} = g^2 \left[ F(s, t, u) + F(t, u, s) + F(u, s, t) \right]$$

$$F(s, t, u) \equiv -\frac{1}{s} \left( 1 - \frac{11}{12} \alpha \left[ \ln(-s/m^2) + c \right] - \frac{1}{2} \alpha \left[ \ln(t/u) \right]^2 \right)$$

which in the high-energy limit can be written as:

$$T = T_0 \left[ 1 - \frac{11}{12} \alpha \left( \ln(s/m^2) + O(m^0) \right) + O(\alpha^2) \right]$$

$$T_0 = -g^2 (s^{-1} + t^{-1} + u^{-1})$$

includes everything without a large logarithm that blows up when $m \to 0$.
Up to this point we assumed that we can isolate individual particles. However there is no energy gap between one-particle states and multi-particle continuum in theories with massless particles:

In this case we cannot distinguish between a one particle state and a state with an extra very low energy (soft) particle that we cannot detect. And so the possibility of having extra soft particles in the process should be included in the calculation!

In addition there could be colinearly moving particles that would look like just one particle in the detector.
Let's account for the possibility that in a given process, described by the amplitude $T$, one of the final state particles splits into two:

![Diagram of particle splitting]

The amplitude for this process can be written as:

$$T_{\text{split}} = ig \frac{-i}{k^2 + m^2} T$$

$$k = k_1 + k_2$$

In the massless limit the propagator can diverge!
If the detector cannot tell whether or not the one particle split into two, then we should add probabilities for the two events (which are in principle distinguishable); we can define an effectively observable squared amplitude as:

$$T_{\text{split}} = ig \frac{-i}{k^2 + m^2} T$$

$$|T|_{\text{obs}}^2 \, dk = |T|^2 \, dk + |T_{\text{split}}|^2 \frac{1}{2} dk_1 \, dk_2 + \ldots$$

Multiplying the second term on the RHS by

$$1 = (2\pi)^{d-1} 2\omega \, \delta^{d-1}(k_1 + k_2 - k) \, \overline{dk}$$

we can remove $\overline{dk}$ and write it as:

$$|T|_{\text{obs}}^2 \equiv |T|^2 \left[ 1 + \frac{g^2}{(k^2 + m^2)^2} \, (2\pi)^{d-1} 2\omega \, \delta^{d-1}(k_1 + k_2 - k) \, \frac{1}{2} dk_1 \overline{dk}_2 + \ldots \right]$$
Note, that splitting of one particle into two gives a contribution comparable to one-loop corrections!

In the massless limit, \( m = 0 \):

\[
k^2 = (k_1 + k_2)^2 = -4 \omega_1 \omega_2 \sin^2(\theta/2)
\]

\[
\tilde{d}k_1 \tilde{d}k_2 \sim (\omega_1^{d-3} \, d\omega_1) \, (\omega_2^{d-3} \, d\omega_2) \, (\sin^{d-3} \theta \, d\theta)
\]

thus, for small \( \theta \):

\[
\frac{\tilde{d}k_1 \tilde{d}k_2}{(k^2)^2} \sim \frac{d\omega_1}{\omega_1^{5-d}} \, \frac{d\omega_2}{\omega_2^{5-d}} \, \frac{d\theta}{\theta^{7-d}}
\]

These integrals would be finite for non-zero \( m \)!
In 6 dimensions we have to worry only about collinear divergences; let’s assume the detector cannot distinguish particles for $\theta$ smaller than some small angle (characteristic for the detector) $\delta$:

$$|T|_{\text{obs}}^2 = |T|^2 \left[ 1 + \frac{g^2}{(k^2 + m^2)^2} \frac{(2\pi)^{d-1}}{2\omega} \delta^{d-1}(k_1 + k_2 - k) \frac{1}{2} \tilde{d}k_1 \tilde{d}k_2 + \ldots \right]$$

$$(2\pi)^5 2\omega \delta^5 (k_1 + k_2 - k) \frac{1}{2} \tilde{d}k_1 \tilde{d}k_2 \rightarrow \frac{\Omega_4}{4(2\pi)^5} \frac{\omega}{\omega_1 \omega_2} |k_1|^4 |k_1| \sin^3 \beta \, d\beta$$

$k_2 = k - k_1$

$\Omega_4 = 2\pi^2$

$\theta = \beta + \gamma$

it is useful to define:

$$\beta = x \theta$$

$$\gamma = (1-x) \theta$$

$0 \leq x \leq 1$

$\theta \leq \delta \ll 1$

everywhere except the propagator we can take $m \to 0$:

$$\omega_1 = |k_1| = (1-x)|k|$$

$$\omega_2 = |k_2| = x|k|$$
expanding the propagator to the leading order in $m$ and theta:

$$k^2 + m^2 \simeq -x(1-x)k^2 \left[ \theta^2 + \left(\frac{m^2}{k^2}\right)f(x) \right]$$

$$f(x) = \frac{(1-x+x^2)}{(x-x^2)^2}$$

changing integration variables, $|k_1|$ and $\beta$ to $x$ and $\theta$ we get:

$$|T|_{obs}^2 \equiv |T|^2 \left[ 1 + \frac{g^2}{(k^2 + m^2)^2} \left( \frac{2\pi}{2}\right)^{d-1} 2\omega \delta^{d-1} (k_1 + k_2 - k) \frac{1}{2} dk_1 dk_2 + \ldots \right]$$

$$\omega_1 = |k_1| = (1-x)|k|$$

$$\beta = x\theta$$

$$\frac{\Omega_4}{4(2\pi)^5} \frac{\omega}{\omega_1 \omega_2} |k_1|^4 d|k_1| \sin^3 \beta d\beta$$

$$|T|_{obs}^2 = |T|^2 \left[ 1 + \frac{g^2 \Omega_4}{4(2\pi)^5} \int_0^1 x(1-x)dx \int_0^\delta \frac{\theta^3 d\theta}{[\theta^2 + \left(\frac{m^2}{k^2}\right)f(x)]^2} + \ldots \right]$$
there is an identical correction from splitting the second particle in the final state; and also the corrections from the two incoming particles are identical; since we have 4 particles the total \( O(\alpha) \) correction due to the failure of our detector to separate two particles with nearly parallel momenta is:

\[
|T|_{\text{obs}}^2 = |T|^2 \left[ 1 + \frac{4}{12} \alpha \left( \ln(\delta^2 k^2/m^2) + c \right) + O(\alpha^2) \right]
\]
combining our result with 1-loop corrections we get:

\[ |T|_{\text{obs}}^2 = |T|^2 \left[ 1 + \frac{4}{12} \alpha \left( \ln(\delta^2 k^2/m^2) + c \right) + O(\alpha^2) \right] \]

\[ T = T_0 \left[ 1 - \frac{11}{12} \alpha \left( \ln(s/m^2) + O(m^0) \right) + O(\alpha^2) \right] \]

\[ k^2 = \frac{1}{4} s \]

\[ |T|_{\text{obs}}^2 = |T_0|^2 \left[ 1 - \frac{11}{6} \alpha \left( \ln(s/m^2) + O(m^0) \right) + O(\alpha^2) \right] \]

\[ \times \left[ 1 + \frac{1}{3} \alpha \left( \ln(\delta^2 s/m^2) + O(m^0) \right) + O(\alpha^2) \right] \]

\[ = |T_0|^2 \left[ 1 - \alpha \left( \frac{3}{2} \ln(s/m^2) + \frac{1}{3} \ln(1/\delta^2) + O(m^0) \right) + O(\alpha^2) \right]. \]

this log is still present, it blows up in the massless limit, we are still doing something wrong; we will discuss it next time.

if we have a very good detector for which this log is not small we need to calculate higher order corrections.
The origin of divergencies in the massless limit is our choice of the renormalization scheme!

For the propagator we have found (at one-loop):

\[
\Pi(k^2) = - \left[ A + \frac{1}{6} \alpha \left( \frac{1}{\varepsilon} + \frac{1}{2} \right) \right] k^2 - \left[ B + \alpha \left( \frac{1}{\varepsilon} + \frac{1}{2} \right) \right] m^2 \\
+ \frac{1}{2} \alpha \int_0^1 dx \, D \ln(D/\mu^2) + O(\alpha^2),
\]

\[
\alpha = g^2/(4\pi)^3
\]

\[
D = x(1-x)k^2 + m^2
\]

\[
\Pi'(k^2) = - \left[ A + \frac{1}{6} \alpha \left( \frac{1}{\varepsilon} + \frac{1}{2} \right) \right] \\
+ \frac{1}{2} \alpha \int_0^1 dx \, x(1-x) \left[ \ln(D/\mu^2) + 1 \right] + O(\alpha^2)
\]

and we imposed:

- \( \Pi(-m^2) = 0 \) to ensure the exact propagator has a pole at \( k^2 = -m^2 \)
- \( \Pi'(-m^2) = 0 \)

but, in the massless limit

satisfied for any A and B!

ill defined!
The problem is that there is no energy gap between one-particle states and multi-particle continuum in theories with massless particles:

Lehmann-Källén form of the exact propagator:

\[
\tilde{\Delta}(k^2) = \frac{1}{k^2 + m^2 - i\epsilon} + \int_{4m^2}^{\infty} ds \rho(s) \frac{1}{k^2 + s - i\epsilon}
\]

In the massless limit:

the pole at \( k^2 = -m^2 \) merges with the branch point at \( k^2 = -4m^2 \)!

There is no isolated pole at \( k^2 = -m^2 \) with residue one, and so the conditions are not meaningful!
The renormalization scheme we have discussed so far:

\[
\Pi(-m^2) = 0 \\
\Pi'(-m^2) = 0
\]

is called the on-shell or OS scheme.

Let's try a different one:

\[
\Pi(k^2) = - \left[ A + \frac{1}{6} \alpha \left( \frac{1}{\xi} + \frac{1}{2} \right) \right] k^2 - \left[ B + \alpha \left( \frac{1}{\xi} + \frac{1}{2} \right) \right] m^2 \\
+ \frac{1}{2} \alpha \int_0^1 dx \ln(D/\mu^2) + O(\alpha^2) ,
\]

\[
A = -\frac{1}{6} \alpha \frac{1}{\xi} + O(\alpha^2) ,
\]

\[
B = -\alpha \frac{1}{\xi} + O(\alpha^2) .
\]

A and B have no finite part

is called the modified minimal-subtraction or \( \overline{\text{MS}} \) scheme.

The correction to the propagator:

\[
\Pi(k^2) = - \left[ A + \frac{1}{6} \alpha \left( \frac{1}{\xi} + \frac{1}{2} \right) \right] k^2 - \left[ B + \alpha \left( \frac{1}{\xi} + \frac{1}{2} \right) \right] m^2 \\
+ \frac{1}{2} \alpha \int_0^1 dx \ln(D/\mu^2) + O(\alpha^2) ,
\]

\[
\Pi_{\overline{\text{MS}}}(k^2) = -\frac{1}{12} \alpha (k^2 + 6m^2) + \frac{1}{2} \alpha \int_0^1 dx \ln(D/\mu^2) + O(\alpha^2)
\]

has a well-defined \( m \to 0 \) limit, but it explicitly depends on the fake parameter \( \mu \)!
Choosing the $\overline{\text{MS}}$ scheme leads to important changes in our calculations:

- the exact propagator will no longer have a pole at $k^2 = -m^2$; by definition the physical mass $m_{\text{ph}}$ of the particle is determined by the location of the pole: $k^2 = -m_{\text{ph}}^2$; the lagrangian parameter $m$ is no longer the same as the physical mass $m_{\text{ph}}$.

- The residue of the pole, $R$, is no longer 1! The LSZ formula must be corrected by multiplying the right hand side by a factor of $R^{-1/2}$ for each external particle because the field $R^{-1/2} \varphi(x)$ is properly normalized to create 1-particle state.

- In the LSZ formula each Klein-Gordon wave operator should be $-\partial^2 + m_{\text{ph}}^2$ when the K-G operator acts on an “external” propagator it cancels the pole and leaves behind the residue $R$.

These changes result in the following Feynman rules:

- assign factors: 
  
  $R^{1/2}$ for each external line
  
  $-i/(k^2 + m^2 - i\epsilon)$ for each internal line with momentum $k$
  
  $iZ_{gg}$ for each vertex
Let’s calculate $m_{ph}$ and $R$:

We have:

$$\Delta_{\text{MS}}(k^2)^{-1} = k^2 + m^2 - \Pi_{\text{MS}}(k^2)$$

by definition

and we find:

$$\Delta_{\text{MS}}(-m_{ph}^2)^{-1} = 0$$

$$k^2 = -m_{ph}^2$$

$$m_{ph}^2 = m^2 - \Pi_{\text{MS}}(-m_{ph}^2)$$

the difference is $O(\alpha)$

$$m_{ph}^2 = m^2 - \Pi_{\text{MS}}(-m^2) + O(\alpha^2)$$
Putting it together we find:

\[ m_{ph}^2 = m^2 - \Pi_{MS}(-m^2) + O(\alpha^2) \]

\[ \Pi_{MS}(k^2) = -\frac{1}{12} \alpha (k^2 + 6m^2) + \frac{1}{2} \alpha \int_0^1 dx D \ln(D/\mu^2) + O(\alpha^2) \]

\[ D = x(1-x)k^2 + m^2 \]

\[ m_{ph}^2 = m^2 - \frac{1}{2} \alpha \left[ \frac{1}{6} m^2 - m^2 + \int_0^1 dx D_0 \ln(D_0/\mu^2) \right] + O(\alpha^2) \]

\[ D_0 = [1-x(1-x)]m^2 \]

and evaluating the integral we get:

\[ m_{ph}^2 = m^2 \left[ 1 + \frac{5}{12} \alpha \left( \ln(\mu^2/m^2) + c' \right) + O(\alpha^2) \right] \]

\[ c' = (34 - 3\pi\sqrt{3})/15 = 1.18 \]

should not depend on the fake parameter \( \mu \)

RHS depends explicitly on \( \mu \)

Solution: \( m \) and \( \alpha \) must depend on \( \mu \)!
We can easily find this dependance:

\[ m_{ph}^2 = m^2 \left[ 1 + \frac{5}{12} \alpha \left( \ln(\mu^2/m^2) + c' \right) + O(\alpha^2) \right] \]

\[
\ln m_{ph} = \ln m + \frac{5}{12} \alpha \left( \ln(\mu/m) + \frac{1}{2} c' \right) + O(\alpha^2)
\]

Does not depend on the fake parameter \( \mu \)

\[ 0 = \frac{d}{d \ln \mu} \ln m_{ph} = \frac{1}{m} \frac{dm}{d \ln \mu} + \frac{5}{12} \alpha + O(\alpha^2) \]

Assumes \( d\alpha/d \ln \mu = O(\alpha^2) \)

We get:

\[
\frac{dm}{d \ln \mu} = \left( -\frac{5}{12} \alpha + O(\alpha^2) \right) m \]

\[ = \gamma_m(\alpha) \quad \text{is called anomalous dimension of the mass parameter} \]
Let's now calculate the residue \( R \):

The residue of a function that has a simple pole is given by:

\[
\text{Res}(f, c) = \lim_{z \to c} (z - c)f(z).
\]

The residue of a function \( f(z) = \frac{g(z)}{h(z)} \) at a simple pole is given by:

\[
\text{Res}(f, c) = \frac{g(c)}{h'(c)}.
\]

\[
\Delta_{\text{MS}}(k^2)^{-1} = k^2 + m^2 - \Pi_{\text{MS}}(k^2)
\]

\[
\Pi_{\text{MS}}(k^2) = -\frac{1}{12} \alpha (k^2 + 6m^2) + \frac{1}{2} \alpha \int_0^1 dx \frac{1}{D} \ln(D/\mu^2) + O(\alpha^2)
\]

\[
D = x(1-x)k^2 + m^2
\]

\[
R^{-1} = 1 - \Pi'_{\text{MS}}(-m_{\text{ph}}^2)
\]

\[
= 1 - \Pi'_{\text{MS}}(-m^2) + O(\alpha^2)
\]

\[
= 1 + \frac{1}{12} \alpha \left( \ln(\mu^2/m^2) + c'' \right) + O(\alpha^2)
\]

\[
c'' = (17 - 3\pi \sqrt{3})/3 = 0.23
\]
The vertex function in the $\overline{\text{MS}}$ scheme:

$$V_3(k_1, k_2, k_3)/g = 1 + \left\{ \alpha \left[ \frac{1}{\varepsilon} + \ln(\mu/m) \right] + C \right\}$$

$$- \frac{1}{2} \alpha \int dF_3 \ln(D/m^2)$$

$$+ O(\alpha^2).$$

$$C = -\alpha \frac{1}{\varepsilon} + O(\alpha^2)$$

$$V_{3,\overline{\text{MS}}}(k_1, k_2, k_3) = g \left[ 1 - \frac{1}{2} \alpha \int dF_3 \ln(D/\mu^2) + O(\alpha^2) \right]$$

$$D = xyk_1^2 + yzk_2^2 + zzk_3^2 + m^2$$

depends explicitly on $\mu$
Now we can repeat the calculation of $\varphi\varphi \rightarrow \varphi\varphi$ in $\varphi^3$ theory in 6 dimensions including all one-loop corrections (in the low-mass limit):

Before we have found:

$$\mathcal{T} = \mathcal{T}_0 \left[ 1 - \frac{11}{12} \alpha \left( \ln\left(\frac{s}{m^2}\right) + O(m^0) \right) + O(\alpha^2) \right]$$

$$\mathcal{T}_0 = -g^2(s^{-1} + t^{-1} + u^{-1})$$

In the $\overline{\text{MS}}$ scheme we get:

$$\mathcal{T} = R^2 \mathcal{T}_0 \left[ 1 - \frac{11}{12} \alpha \left( \ln\left(\frac{s}{\mu^2}\right) + O(m^0) \right) + O(\alpha^2) \right]$$

$$R^{-1} = 1 + \frac{1}{12} \alpha \left( \ln\left(\frac{\mu^2}{m^2}\right) + c'' \right) + O(\alpha^2)$$

$$\mathcal{T} = \mathcal{T}_0 \left[ 1 - \alpha \left( \frac{11}{12} \ln\left(\frac{s}{\mu^2}\right) + \frac{1}{6} \ln\left(\frac{\mu^2}{m^2}\right) + O(m^0) \right) + O(\alpha^2) \right]$$
To get an observable amplitude-squared we have to correct for an imperfect detector; we found:

\[ |T|_{\text{obs}}^2 = |T|^2 \left[ 1 + \frac{1}{3} \alpha \left( \ln(\delta^2 s/m^2) + O(m^0) \right) + O(\alpha^2) \right] \]

In the \( \overline{\text{MS}} \) scheme we get:

\[ T = T_0 \left[ 1 - \alpha \left( \frac{11}{12} \ln(s/\mu^2) + \frac{1}{6} \ln(\mu^2/m^2) + O(m^0) \right) + O(\alpha^2) \right] \]

\[ |T|_{\text{obs}}^2 = |T_0|^2 \left[ 1 - \alpha \left( \frac{3}{2} \ln(s/\mu^2) + \frac{1}{3} \ln(1/\delta^2) + O(m^0) \right) + O(\alpha^2) \right] \]

has a well-defined \( m \to 0 \) limit, but it explicitly depends on the fake parameter \( \mu \)!

should not depend on the fake parameter \( \mu \)

Solution: \( \alpha \) must depend on \( \mu \)!
\[ |T|_{\text{obs}}^2 = |T_0|^2 \left[ 1 - \alpha \left( \frac{3}{2} \ln(s/\mu^2) + \frac{1}{3} \ln(1/\delta^2) + O(m^0) \right) + O(\alpha^2) \right] \]

should not depend on the fake parameter \( \mu \)

\[ \ln |T|_{\text{obs}}^2 = C_1 + 2 \ln \alpha + 3\alpha \left( \ln \mu + C_2 \right) + O(\alpha^2) \]

\[ |T_0|^2 = O(g^4) = O(\alpha^2) \]

we find:

\[ 0 = \frac{d}{d \ln \mu} \ln |T|_{\text{obs}}^2 \]

\[ = \frac{2}{\alpha} \frac{d\alpha}{d \ln \mu} + 3\alpha + O(\alpha^2) \]

\[ \frac{d\alpha}{d \ln \mu} = -\frac{3}{2} \alpha^2 + O(\alpha^3) \]

\[ = \beta(\alpha) \text{ is called the beta function} \]
\[|T|_{\text{obs}}^2 = |T_0|^2 \left[ 1 - \alpha \left( \frac{3}{2} \ln(s/\mu^2) + \frac{1}{3} \ln(1/\delta^2) + O(m^0) \right) + O(\alpha^2) \right] \]

Note: we can choose any \( \mu \) we might like, but to avoid large logs we should choose \( \mu^2 \sim s \)!

To compare results at different energies we can solve the differential equation:

\[ \frac{d\alpha}{d \ln \mu} = -\frac{3}{2} \alpha^2 + O(\alpha^3) \]

\[ \alpha(\mu_2) = \frac{\alpha(\mu_1)}{1 + \frac{3}{2} \alpha(\mu_1) \ln(\mu_2/\mu_1)} \]

decreases as \( \mu \) increases

theories with this property are said to be asymptotically free
(the tree level approximation becomes better and better at higher and higher energies)
(at lower and lower energies the theory becomes more and more strongly coupled)
The renormalization group

We are going to derive, in a systematic way, how lagrangian parameters and other object that are not directly measurable vary with $\mu$:

Let's consider the $\varphi^3$ theory in $d = 6 - \varepsilon$ dimensions:

$$
\mathcal{L} = -\frac{1}{2}Z_\varphi \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2}Z_m m^2 \varphi^2 + \frac{1}{6}Z_g g \tilde{\mu}^{\varepsilon/2} \varphi^3 + Y \varphi
$$

renormalized fields and parameters (in $\overline{MS}$ scheme)

The same theory can be also written as:

$$
\mathcal{L} = -\frac{1}{2}\partial^\mu \varphi_0 \partial_\mu \varphi_0 - \frac{1}{2}m_0^2 \varphi_0^2 + \frac{1}{6}g_0 \varphi_0^3 + Y_0 \varphi_0
$$

bare fields and parameters must be independent of $\mu$!

The dictionary:

$$
\begin{align*}
\varphi_0(x) &= Z_\varphi^{1/2} \varphi(x), \\
m_0 &= Z_\varphi^{-1/2} Z_m^{1/2} m, \\
g_0 &= Z_\varphi^{-3/2} Z_g g \tilde{\mu}^{\varepsilon/2}, \\
Y_0 &= Z_\varphi^{-1/2} Y.
\end{align*}
$$
In the $\overline{\text{MS}}$ scheme we choose the $Z$s to cancel the divergent parts of loop integrals and so in general:

$$Z_{\varphi} = 1 + \sum_{n=1}^{\infty} \frac{a_n(\alpha)}{\varepsilon^n},$$

$$Z_{m} = 1 + \sum_{n=1}^{\infty} \frac{b_n(\alpha)}{\varepsilon^n},$$

$$Z_{g} = 1 + \sum_{n=1}^{\infty} \frac{c_n(\alpha)}{\varepsilon^n},$$

functions of $\alpha = g^2/(4\pi)^3$.

at one loop we found:

$$a_1(\alpha) = -\frac{1}{6}\alpha + O(\alpha^2),$$

$$b_1(\alpha) = -\alpha + O(\alpha^2),$$

$$c_1(\alpha) = -\alpha + O(\alpha^2),$$

for $n \geq 2$ all $a_n(\alpha)$, $b_n(\alpha)$ and $c_n(\alpha)$ are at least $O(\alpha^2)$!

$$A = Z_{\varphi} - 1, \quad B = Z_{m} - 1 \quad Z_{g} = 1 + C$$

$$\Pi(k^2) = -\left[ A + \frac{1}{6}\alpha\left(\frac{1}{\varepsilon} + \frac{1}{2}\right) \right] k^2 - \left[ B + \alpha\left(\frac{1}{\varepsilon} + \frac{1}{2}\right) \right] m^2$$

$$+ \frac{1}{2}\alpha \int_0^1 dx \ln(D/\mu^2) + O(\alpha^2),$$

$$V_3(k_1, k_2, k_3)/g = 1 + \left\{ \alpha\left[ \frac{1}{\varepsilon} + \ln(\mu/m) \right] + C \right\}$$

$$- \frac{1}{2}\alpha \int dF_3 \ln(D/m^2)$$

$$+ O(\alpha^2).$$
Let's derive consequences of bare gauge coupling being $\mu$ independent:

Let's define:

$$\alpha_0 \equiv g_0^2/(4\pi)^3 = Z_g^2 Z_\varphi^{-3} \tilde{\mu}^\varepsilon \alpha$$

$$G(\alpha, \varepsilon) \equiv \ln(Z_g^2 Z_\varphi^{-3})$$

$$G(\alpha, \varepsilon) = \sum_{n=1}^{\infty} \frac{G_n(\alpha)}{\varepsilon^n}$$

in particular: $G_1(\alpha) = 2c_1(\alpha) - 3a_1(\alpha)$

$$a_1(\alpha) = -\frac{1}{6}\alpha + O(\alpha^2),$$

$$b_1(\alpha) = -\alpha + O(\alpha^2),$$

$$c_1(\alpha) = -\alpha + O(\alpha^2),$$

$$\ln \alpha_0 = G(\alpha, \varepsilon) + \ln \alpha + \varepsilon \ln \tilde{\mu}$$

$$g_0 = Z_\varphi^{-3/2} Z_g \tilde{\mu}^\varepsilon / 2$$

$$Z_\varphi = 1 + \sum_{n=1}^{\infty} \frac{a_n(\alpha)}{\varepsilon^n}$$

$$Z_g = 1 + \sum_{n=1}^{\infty} \frac{c_n(\alpha)}{\varepsilon^n}$$
does not depend on the fake parameter $\mu$

\[
\ln \alpha_0 = G(\alpha, \varepsilon) + \ln \alpha + \varepsilon \ln \tilde{\mu}
\]

\[
0 = \frac{d}{d \ln \mu} \ln \alpha_0
\]

\[
= \frac{\partial G(\alpha, \varepsilon)}{\partial \alpha} \frac{d\alpha}{d \ln \mu} + \frac{1}{\alpha} \frac{d\alpha}{d \ln \mu} + \varepsilon
\]

regrouping the terms we get:

\[
0 = \left(1 + \frac{\alpha G'_1(\alpha)}{\varepsilon} + \frac{\alpha G'_2(\alpha)}{\varepsilon^2} + \ldots\right) \frac{d\alpha}{d \ln \mu} + \varepsilon \alpha
\]

we can find the solution by matching terms at with given power of $\varepsilon$:

\[
\frac{d\alpha}{d \ln \mu} = -\varepsilon \alpha + \beta(\alpha)
\]

should be finite as $\varepsilon \to 0$

matching $O(\varepsilon)$ terms:

\[
\beta(\alpha) = \alpha^2 G'_1(\alpha)
\]

matching terms with higher powers of $1/\varepsilon$ leads to conditions among $G'_n(\alpha)$'s, e.g. matching the $O(\varepsilon^{-1})$ we get $G'_2(\alpha) = \alpha G'_1(\alpha)^2$, ...

... can be checked order by order in perturbation theory!
Putting it together we find:

\[
\frac{d\alpha}{d \ln \mu} = -\varepsilon \alpha + \beta(\alpha)
\]

\[
\beta(\alpha) = \alpha^2 G'(\alpha)
\]

\[
G_1(\alpha) = 2c_1(\alpha) - 3a_1(\alpha)
\]

\[
= -\frac{3}{2}\alpha + O(\alpha^2).
\]

\[
\beta(\alpha) = -\frac{3}{2}\alpha^2 + O(\alpha^3)
\]

the same result as we obtained before from requiring that a particular cross section is independent of $\mu$!
Let's repeat the same thing for the bare mass parameter:

Let's define:

\[
M(\alpha, \varepsilon) \equiv \ln(Z_m^{1/2}Z_\phi^{-1/2}) = \sum_{n=1}^{\infty} \frac{M_n(\alpha)}{\varepsilon^n}.
\]

in particular, we find:

\[
M_1(\alpha) = \frac{1}{2}b_1(\alpha) - \frac{1}{2}a_1(\alpha) = -\frac{5}{12}\alpha + O(\alpha^2).
\]

\[
a_1(\alpha) = -\frac{1}{6}\alpha + O(\alpha^2), \quad b_1(\alpha) = -\alpha + O(\alpha^2), \quad c_1(\alpha) = -\alpha + O(\alpha^2),
\]

taking the log:

\[
\ln m_0 = M(\alpha, \varepsilon) + \ln m
\]
\[ \ln m_0 = M(\alpha, \varepsilon) + \ln m \]

does not depend on the fake parameter \( \mu \)

\[ 0 = \frac{d}{d \ln \mu} \ln m_0 \]

\[ = \frac{\partial M(\alpha, \varepsilon)}{\partial \alpha} \frac{d\alpha}{d \ln \mu} + \frac{1}{m} \frac{dm}{d \ln \mu} . \]

\[ = \frac{\partial M(\alpha, \varepsilon)}{\partial \alpha} (-\varepsilon \alpha + \beta(\alpha)) + \frac{1}{m} \frac{dm}{d \ln \mu} . \]

\[ \frac{d\alpha}{d \ln \mu} = -\varepsilon \alpha + \beta(\alpha) \]

rearranging, we find

\[ \frac{1}{m} \frac{dm}{d \ln \mu} = (\varepsilon \alpha - \beta(\alpha)) \sum_{n=1}^{\infty} \frac{M_n'(\alpha)}{\varepsilon^n} \]

should be finite as \( \varepsilon \to 0 \)

\[ = \alpha M'_1(\alpha) + \ldots , \]

terms with powers of \( 1/\varepsilon \)

these terms must all be zero!
For the anomalous dimension of the mass we get:

\[
\gamma_m(\alpha) \equiv \frac{1}{m} \frac{dm}{d \ln \mu} = \left(\varepsilon \alpha - \beta(\alpha)\right) \sum_{n=1}^{\infty} \frac{M'_n(\alpha)}{\varepsilon^n} = \alpha M'_1(\alpha)
\]

\[
M_1(\alpha) = \frac{1}{2} b_1(\alpha) - \frac{1}{2} a_1(\alpha) = -\frac{5}{12} \alpha + O(\alpha^2).
\]

For the anomalous dimension of the mass we get:

\[
\gamma_m(\alpha) = \alpha M'_1(\alpha)
\]

\[
= -\frac{5}{12} \alpha + O(\alpha^2)
\]

the same result as we obtained before from requiring that a particular cross section is independent of \( \mu \)!
We can repeat the same procedure for the propagator in the $\overline{\text{MS}}$ scheme:

\[
\tilde{\Delta}(k^2) = i \int d^6x \ e^{ikx} \langle 0 | T\varphi(x)\varphi(0) | 0 \rangle
\]

first, we obtain the bare propagator:

\[
\tilde{\Delta}_0(k^2) = i \int d^6x \ e^{ikx} \langle 0 | T\varphi_0(x)\varphi_0(0) | 0 \rangle
\]

\[
\varphi_0(x) = Z_\varphi^{1/2} \varphi(x)
\]

\[
\tilde{\Delta}_0(k^2) = Z_\varphi \tilde{\Delta}(k^2)
\]

should be independent of $\mu$

\[
0 = \frac{d}{d \ln \mu} \ln \tilde{\Delta}_0(k^2)
\]

\[
= \frac{d \ln Z_\varphi}{d \ln \mu} + \frac{d}{d \ln \mu} \ln \tilde{\Delta}(k^2)
\]

\[
= \frac{d \ln Z_\varphi}{d \ln \mu} + \frac{1}{\tilde{\Delta}(k^2)} \left( \frac{\partial}{\partial \ln \mu} + \frac{d\alpha}{d \ln \mu} \frac{\partial}{\partial \alpha} + \frac{dm}{d \ln \mu} \frac{\partial}{\partial m} \right) \tilde{\Delta}(k^2)
\]
0 = \frac{d \ln Z_\varphi}{d \ln \mu} + \frac{1}{\tilde{\Delta}(k^2)} \left( \frac{\partial}{\partial \ln \mu} + \frac{d \alpha}{d \ln \mu} \frac{\partial}{\partial \alpha} + \frac{dm}{d \ln \mu} \frac{\partial}{\partial m} \right) \tilde{\Delta}(k^2)

Z_\varphi = 1 + \sum_{n=1}^{\infty} \frac{a_n(\alpha)}{\varepsilon^n}

\ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots

\ln Z_\varphi = \frac{a_1(\alpha)}{\varepsilon} + \frac{a_2(\alpha) - \frac{1}{2} a_1^2(\alpha)}{\varepsilon^2} + \ldots

\frac{d \ln Z_\varphi}{d \ln \mu} = \frac{\partial \ln Z_\varphi}{\partial \alpha} \frac{d \alpha}{d \ln \mu}

= \left( \frac{a_1'(\alpha)}{\varepsilon} + \ldots \right) \left( -\varepsilon \alpha + \beta(\alpha) \right)

= -\alpha a_1'(\alpha) + \ldots ,

should be finite as \( \varepsilon \to 0 \)

terms with powers of \( 1/\varepsilon \)

despite these terms must all be zero!
Defining the anomalous dimension of the field:

\[ \gamma_\varphi(\alpha) = \frac{1}{2} \frac{d \ln Z_\varphi}{d \ln \mu} \]

\[ \gamma_\varphi(\alpha) = -\frac{1}{2} \alpha a_1'(\alpha) = +\frac{1}{12} \alpha + O(\alpha^2) \]

We obtain:

\[ \left( \frac{\partial}{\partial \ln \mu} + \beta(\alpha) \frac{\partial}{\partial \alpha} + \gamma_m(\alpha) m \frac{\partial}{\partial m} + 2 \gamma_\varphi(\alpha) \right) \tilde{\Delta}(k^2) = 0 \]

Callan-Symanzik equation for the propagator