Schwinger-Dyson equations

The path integral

\[ Z(J) = \int D\varphi \, e^{i[S + \int d^4y J_a \varphi_a]} \]

doesn’t change if we change variables \( \varphi_a(x) \to \varphi_a(x) + \delta \varphi_a(x) \) assuming the measure \( D\varphi \) is invariant under the change of variables thus we have:

\[
0 = \delta Z(J) = i \int D\varphi \, e^{i[S + \int d^4y J_a \varphi_a]} \int d^4x \left( \frac{\delta S}{\delta \varphi_a(x)} + J_a(x) \right) \delta \varphi_a(x) \]

taking n functional derivatives with respect to \( J_a(x) \) and setting \( J = 0 \) we get:

\[
0 = \int D\varphi \, e^{iS} \int d^4x \left[ i \frac{\delta S}{\delta \varphi_a(x)} \varphi_a(x_1) \ldots \varphi_a(x_n) + \sum_{j=1}^{n} \varphi_a(x_1) \ldots \delta_{a_j} \delta^4(x-x_j) \ldots \varphi_a(x_n) \right] \delta \varphi_a(x)
\]

A comment on functional derivative:

\[
\frac{\delta \varphi_b(y)}{\delta \varphi_a(x)} = \delta_{ba} \delta^4(y-x)
\]

\[
\delta \varphi_b(y) = \delta_{ba} \delta^4(y-x) \delta \varphi_a(x)
\]

\[
\delta \varphi_b(y) = \sum_{a} \int d\varphi \, \delta_{ba} \delta^4(y-x) \delta \varphi_a(x)
\]

Similarly:

\[
\frac{\delta F(\varphi_a(x))}{\delta \varphi_b(y)} = \frac{\partial F}{\partial \varphi_a(x)} \frac{\delta \varphi_a(x)}{\delta \varphi_b(y)} = \frac{\partial F}{\partial \varphi_a(x)} \delta_{ba} \delta^4(y-x)
\]

\[
\delta F(\varphi_a(x)) = \sum_{b} \int dy^4 \, \frac{\partial F}{\partial \varphi_b(y)} \delta_{ba} \delta^4(y-x) \delta \varphi_b(y) = \frac{\partial F}{\partial \varphi_a(x)} \delta \varphi_a(x)
\]

or, for the action:

\[
\delta S = \int dx^4 \frac{\delta S}{\delta \varphi_a(x)} \delta \varphi_a(x)
\]
Ward-Takahashi identity:

For a theory with a continuous symmetry we can consider transformations that result in $\delta L = 0$:

$$0 = \int d^4\varphi \sum_{j=1}^{n} (0|T\varphi_{a_j}(x_1)\ldots\varphi_{a_n}(x_n)|0) + i \sum_{j=1}^{n} (0|T\varphi_{a_j}(x_1)\ldots \delta \varphi_{a_j}(x)\delta^4(x-x_j)\ldots \varphi_{a_n}(x_n)|0)$$

Ward-Takahashi identity

summing over $a$ and dropping the integral over $d^4x$,

thus, conservation of the Noether current holds in the quantum theory, with the current inside a correlation function, up to contact terms (that depend on the infinitesimal transformation).

Ward identities in QED

When discussing QED we used the result that a scattering amplitude for a process that includes an external photon with momentum $k^\mu$ should satisfy (as a result of Ward identity):

$$k^\mu M_\mu = 0$$

$$T = \varepsilon^\mu_\lambda(k)M_\mu$$

we used it, for example, to obtain a simple formula for the photon polarization sum that we needed for calculations of cross sections:

$$\sum_{\lambda = \pm} \varepsilon^{\mu\lambda}(k)\varepsilon^{\rho\lambda}(k) \rightarrow g^{\mu\rho}$$

now we are going to prove it!

To simplify the discussion let’s treat all particles in the LSZ formula as outgoing (incoming particles have $k_i^0 < 0$):

$$\langle f|i = \int d^4x e^{-ikx} \langle 0|T\varphi(x_1)\ldots|0\rangle$$

it can be also written as:

$$\langle f|i = \lim_{k^2 \rightarrow -m^2} \langle 0|T\varphi(k_1)\ldots|0\rangle$$

$$\varphi(k) = i \int d^4x e^{-ikx}\varphi(x)$$

we do not fix $k^2 = -m^2$

must include an overall energy-momentum delta function

$$\langle 0|T\varphi(k_1)\ldots|0\rangle = (2\pi)^4\delta^4(\sum_i k_i)F(k_i^2, k_i.k_j)$$

we used it, for example, to obtain a simple formula for the photon polarization sum that we needed for calculations of cross sections:

$$\sum_{\lambda = \pm} \varepsilon^{\mu\lambda}(k)\varepsilon^{\rho\lambda}(k) \rightarrow g^{\mu\rho}$$

now we are going to prove it!

(residue of the pole

multivariable pole

corrections that do not have this multivariable pole do not contribute to $iT$ !

(for simplicity we work with scalar fields, but the same applies to fields of any spin)
near $k_i^2 = -m^2$ we can write:

$$\mathcal{F}(k_i^2, k_i \cdot k_j) = \frac{iT}{(k_i^2 + m^2) \ldots (k_n^2 + m^2)} + \text{nonsingular}$$

residue of the pole

multivariable pole

contributions that do not have this multivariable pole do not contribute to $iT$!

this means that in Schwinger-Dyson equations:

$$\left(0|T \frac{\delta S}{\delta \phi_a(x)} \phi_{a_1}(x_1) \ldots \phi_{a_n}(x_n)|0\right) = i \sum_{j=1}^n \left(0|T \phi_{a_1}(x_1) \ldots \phi_{a_{j-1}}(x_{j-1}) \delta^4(x-x_j) \ldots \phi_{a_n}(x_n)|0\right)$$

classical eq.
of motion

$\delta S/\delta \phi_a(x) = 0$

contact terms:

$\delta^4(x-x_2)$

in the momentum space a contact term is a function of $k_1 + k_2$; it doesn't have the right pole structure to contribute!

contact terms in a correlation function $F$ do not contribute to the scattering amplitude!

Let's consider a scattering process in QED with an external photon: with momentum $k^\mu$

the LSZ formula:

$$\langle f | i \rangle = i \epsilon^{\mu} \int d^4x e^{-ikx} (-\partial^2) \ldots \langle 0| T A_\mu(x) \ldots |0\rangle$$

the classical equation of motion in the Lorentz gauge:

$$-Z_3 \partial^2 A_\mu = \frac{\partial L}{\partial A_\mu} + J^\mu = 0$$

thus we find:

$$\langle f | i \rangle = i Z_3^{-1} Z_1 \epsilon^{\mu} \int d^4x e^{-ikx} \ldots \langle 0| T j_\mu(x) \ldots |0\rangle + \text{contact terms}$$

contact terms cannot generate the proper singularities and thus these do not contribute to the scattering amplitude!

Finally, we will derive another consequence of Ward identity that we used:

$$Z_1 = Z_2$$

consider the correlation function:

$$C_{\alpha \beta}(k, p', p) = i Z_1 \int d^4x d^4y d^4z e^{ikz-i p'y+ipz} \langle 0| T j_\mu(x) \Psi_\alpha(y) \Psi_\beta(z)|0\rangle$$

adds a vertex $L_1 = Z_1 e^2 \bar{\Phi} \Phi$

in the momentum space:

$$C_{\alpha \beta}(k, p', p) = (2\pi)^4 \delta^4(k+p-p') \left[ \frac{1}{2} \bar{S}(p') i \Gamma^{\mu}(p', p) \right]_{\alpha \beta}$$

later we will use:

$$k_\mu C_{\alpha \beta}^{\mu}(k, p', p) = -i(2\pi)^4 \delta^4(k+p-p') \left[ \bar{S}(p') k_\mu \Gamma^{\mu}(p', p) S(p) \right]_{\alpha \beta}$$

exact propagator

exact 1PI vertex function
integrate by parts

multiply by \( k^\mu \)

\[ i k^\mu \rightarrow \partial_\mu e^{ikx} \]

integrate by parts

\[ k^\mu C_{\alpha\beta}(k, p', p) = -\int d^4x \, d^4y \, d^4z \, e^{ikx - ip'y + ipz} \partial_\mu(0|Tj^\mu(x)\Psi_\alpha(y)\overline{\Psi}_\beta(z)|0) \]

\[ \partial_\mu(0|Tj^\mu(x)\Psi_\alpha(y)\overline{\Psi}_\beta(z)|0) \]

\[ = -\int d^4x \, d^4y \, d^4z \, e^{ikx - ip'y + ipz} \partial_\mu(0|Tj^\mu(x)\Psi_\alpha(y)\overline{\Psi}_\beta(z)|0) \]

\[ \partial_\mu(0|Tj^\mu(x)\Psi_\alpha(y)\overline{\Psi}_\beta(z)|0) \]

The Ward identity:

\[ \partial_\mu(0|Tj^\mu(x)\phi_\alpha(x_1)\ldots\phi_\alpha(x_n)|0) \]

\[ = i \sum_{j=1}^n \int d^4x \, d^4y \, d^4z \, e^{ikx - ip'y + ipz} \partial_\mu(0|Tj^\mu(x)\Psi_\alpha(y)\overline{\Psi}_\beta(z)|0) \]

\[ \partial_\mu(0|Tj^\mu(x)\Psi_\alpha(y)\overline{\Psi}_\beta(z)|0) \]

\[ \partial_\mu(0|Tj^\mu(x)\Psi_\alpha(y)\overline{\Psi}_\beta(z)|0) \]

\[ J^\mu = Z_2 e^{i\phi} \Psi = Z_2 j^\mu \]

\[ J^\mu = Z_2 e^{i\phi} \Psi = Z_2 j^\mu \]

\[ = \frac{1}{i} \int \frac{d^4q}{(2\pi)^4} \, e^{iq(x-y)} \mathcal{S}(q)_{\alpha\beta} \]

\[ k^\mu C_{\alpha\beta}(k, p', p) = -iZ_2^{-1} Z_1(2\pi)^4 \delta^4(k+p-p') \left[ e\mathcal{S}(p) - e\mathcal{S}(p') \right]_{\alpha\beta} \]

before we found:

\[ k^\mu C_{\alpha\beta}(k, p', p) = -i(2\pi)^4 \delta^4(k+p-p') \left[ e\mathcal{S}(p) - e\mathcal{S}(p') \right]_{\alpha\beta} \]

thus we get:

\[ (p'-p)_\mu \mathcal{S}(p)p\mathcal{V}(p', p) = Z_2^{-1} Z_1 e \left[ \mathcal{S}(p) - \mathcal{S}(p') \right] \]

or:

\[ (p'-p)_\mu \mathcal{V}(p', p) = Z_2^{-1} Z_1 e \left[ \mathcal{S}(p')^{-1} - \mathcal{S}(p)^{-1} \right] \]

in the \( \overline{\text{MS}} \) scheme

\[ Z_1 = Z_2 \]

finite

since the divergent pieces must cancel, and there are only divergent pieces

In the OS scheme, near \((p' - p)^2 = 0\), \( p^2 = p'^2 = -m^2 \), we have:

\[ V^\mu(p', p) = e^{i\phi^\mu} \mathcal{S}(p) = p + m \]

and so we get \( Z_1 = Z_2 \)

The Ward identity means that the kinetic term \( iZ_2 \overline{\psi} \Psi \) and the interaction term \( Z_1 e\overline{\psi} A \psi \) are renormalized in the same way and so they can be combined into covariant derivative term \( iZ_2 \overline{\psi} \gamma^\mu \psi \) (which we could have guessed from gauge invariance)!

\[ D^\mu = \partial^\mu - ieA^\mu \]

Formal development of fermionic PI

based on S-44

We want to derive the fermionic path integral formula (that we previously postulated by analogy with the path integral for a scalar field):

\[ Z_0(\bar{\eta}, \eta) = \int \mathcal{D}\Psi \, \mathcal{D}\overline{\Psi} \, \exp \left[ i \int d^4x \overline{\Psi}(i\overline{\partial} - m)\Psi + \bar{\eta}\Psi + \bar{\Psi}\eta \right] \]

\[ = \exp \left[ i \int d^4x \, d^4y \, \bar{\eta}(x)S(x - y)\eta(y) \right], \]

\[ (-i\overline{\partial}_x + m)S(x - y) = \delta^4(x - y) \]
Let's define a set of anticommuting numbers or Grassmann variables:

\[ \{ \psi_i, \psi_j \} = 0 \quad i = 1, \ldots, n \]

for \( n = 1 \) we have just one number \( \psi \) with \( \psi^2 = 0 \).

We define a function of \( \psi \) by a Taylor expansion:

\[ f(\psi) = a + \psi b \]

the order is important!

if \( f \) itself is commuting then \( b \) has to be an anticommuting number:

\[ \{ b, b \} = \{ b, \psi \} = 0 \]

and we have:

\[ f(\psi) = a + \psi b = a - b \psi \]

Let's define the left derivative of \( f(\psi) \) with respect to \( \psi \) as:

\[ \partial_\psi f(\psi) = +b \quad f(\psi) = a + \psi b \]

Similarly, let's define the left derivative of \( f(\psi) \) with respect to \( \psi \) as:

\[ f(\psi) \partial_\psi = -b \]

We define the definite integral with the same properties as those of an integral over a real variable; namely linearity and invariance under shifts:

\[ \int_{-\infty}^{+\infty} dx \ c f(x) = c \int_{-\infty}^{+\infty} dx \ f(x) \quad \int_{-\infty}^{+\infty} dx \ f(x + a) = \int_{-\infty}^{+\infty} dx \ f(x) \]

The only possible nontrivial definition (up to an overall numerical factor) is:

\[ \int d\psi \ f(\psi) = b \]

Let's generalize this to \( n > 1 \), we have:

\[ f(\psi) = a + \psi b_i + \frac{1}{2} \psi_i \psi_j c_{i,j} + \ldots + \frac{1}{n!} \psi_i \ldots \psi_n \ d_{i_1 \ldots i_n} \]

all indices summed over completely antisymmetric on exchange of any two indices

Let's define the left derivative of \( f(\psi) \) with respect to \( \psi_j \) as:

\[ \frac{\partial}{\partial \psi_j} f(\psi) = b_j + \psi_i c_{j,i} + \ldots + \frac{1}{(n-1)!} \psi_{i_2} \ldots \psi_{i_n} d_{j i_2 \ldots i_n} \]

and similarly for the right derivative...

To define (linear and shift invariant) integral note that:

\[ d_{i_1 \ldots i_n} = \delta \varepsilon_{i_1 \ldots i_n} \]

just a number (if \( n \) is even)

Levi-Civita symbol, \( \varepsilon_{1 \ldots n} = +1 \)

the only consistent definition of the integral is:

\[ \int d^n \psi \ f(\psi) = d \]

alternatively we could write the differential in terms of individual differentials:

\[ d^n \psi = d \psi_n \ldots d \psi_1 \]

and use

\[ \int d\psi_i = 0 \quad \int d\psi_i \psi_j = \delta_{ij} \]

to derive the result above.
Consider a linear change of variable:

\[ \psi_i = J_{ij} \psi_j \]

then we have:

\[ f(\psi) = a + \ldots + \frac{1}{n!} (J_{i_1 j_1} \psi_{j_1}) \ldots (J_{i_n j_n} \psi_{j_n}) \varepsilon_{i_1 \ldots i_n} d \]

integrating over \( d^p \psi \) we get \( \int d^p \psi f(\psi) = d \)

Recall, for integrals over real numbers with \( x_i = J_{ij} x_j \) we have:

\[ \int d^n x f(x) = (\text{det } J)^{-1} \int d^n x' f(x) \]

We are interested in gaussian integrals of the form:

\[ \int d^n \psi \exp\left( \frac{1}{2} \psi^T M \psi \right) \]

We finally get:

\[ \int d^2 \psi \exp\left( \frac{1}{2} \psi^T M \psi \right) = m \]

For larger (even) \( n \) we can bring a complex antisymmetric matrix to a block-diagonal form:

\[ U^T M U = \begin{pmatrix} 0 & +m_1 \\ -m_1 & 0 \end{pmatrix} \]

a unitary matrix

we will later need:

\( (\text{det } U)^2 (\text{det } M) = \prod_{l=1}^{n/2} m_l^2 \)

taking:

\[ \psi_i = U_{ij} \psi'_j \]

we have:

\[ \int d^p \psi f(\psi) = (\text{det } J)^{-1} \int d^p \psi' f(\psi) \]

\[ \int d^p \psi \exp\left( \frac{1}{2} \psi^T M \psi \right) = (\text{det } U)^{-1} \prod_{l=1}^{n/2} \int d^2 \psi_{li} \exp\left( \frac{1}{2} \psi_{li}^T M_{ii} \psi_{li} \right) \]

we drop primes

we will later need:

\[ \int d^2 \psi \exp\left( \frac{1}{2} \psi^T M \psi \right) = m \]

using the result for \( n = 2 \) we get:

\[ \int d^2 \psi \exp\left( \frac{1}{2} \psi^T M \psi \right) = (\text{det } U)^{-1} \prod_{l=1}^{n/2} m_l \]

we finally get:

\[ \int d^p \psi \exp\left( \frac{1}{2} \psi^T M \psi \right) = (\text{det } M)^{1/2} \]

Recall, for integrals over real numbers we have:

\[ \int d^x \exp\left( -\frac{1}{2} x^T M x \right) = (2\pi)^{n/2} (\text{det } M)^{-1/2} \]

\( \int_{-\infty}^{\infty} e^{-ix} dx = \sqrt{\pi} \)
Let’s define complex Grassmann variables:
\[ \chi = \frac{1}{\sqrt{2}} (\psi_1 + i\psi_2) \]
\[ \bar{\chi} = \frac{1}{\sqrt{2}} (\psi_1 - i\psi_2) \]
we can invert this to get:
\[ \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} \bar{\chi} \\ \chi \end{pmatrix} \]
\[ \delta \psi = d\psi_2 d\psi_1 = (-i)^{-1} d\chi d\bar{\chi} \]
also since \( \psi_1 \psi_2 = -i \bar{\chi} \chi \) we have:
\[ \int d\chi d\bar{\chi} \bar{\chi} \chi = (-i)(-i)^{-1} \int d\psi_2 d\psi_1 \psi_1 \psi_2 = 1 \]

A function can be again defined by a Taylor expansion:
\[ f(\chi, \bar{\chi}) = a + \chi b + \bar{\chi} c + \bar{\chi} \chi d \]
the integral is:
\[ \int d\chi d\bar{\chi} f(\chi, \bar{\chi}) = d \]
in particular:
\[ \int d\chi d\bar{\chi} \exp(m \bar{\chi} \chi) = m \]

Let’s consider \( n \) complex Grassmann variables and their conjugates:
define:
\[ d^n \chi d^n \bar{\chi} \equiv d\chi_n d\bar{\chi}_n \ldots d\chi_1 d\bar{\chi}_1 \]
under a change of variable:
\[ \chi_i = J_{ij} \chi_j \quad \bar{\chi}_i = \bar{K}_{ij} \bar{\chi}_j \]
\[ K_{ij} = J_{ij}^* \not\text{not important} \]
(the integral doesn’t care whether \( \bar{\chi}_i \) is the complex conjugate of \( \chi_i \))
we have:
\[ d^n \chi d^n \bar{\chi} = (\det J)^{-1} (\det K)^{-1} d^n \chi' d^n \bar{\chi}' \]
we want to evaluate:
\[ \int d^n \chi d^n \bar{\chi} \exp(\chi^\dagger M \chi) \]

\[ \chi = U \chi' \quad \chi^\dagger = \chi'^\dagger V \]
\[ \int d^n \chi d^n \bar{\chi} \exp(\chi^\dagger M \chi) \]
under such a change of variable we get:
\[ \int d^n \chi d^n \bar{\chi} \exp(\chi^\dagger M \chi) = (\det U)^{-1} (\det V)^{-1} \prod_{i=1}^n \int d\chi_i d\bar{\chi}_i \exp(m_i \bar{\chi}_i \chi_i) \]
\[ = (\det U)^{-1} (\det V)^{-1} \prod_{i=1}^n \delta(\chi_i) \]
\[ = \det M . \]
we drop primes
\[ \int d^n \chi d^n \bar{\chi} \exp(m \bar{\chi} \chi) = m \]

Analogous integral for commuting complex variable
\[ z_i = (x_i + i y_i)/\sqrt{2} \]
\[ \bar{z}_i = (x_i - i y_i)/\sqrt{2} \]
\[ d^m z d^m \bar{z} = d^m z d^m \bar{z} \]
\[ d^n z d^n \bar{z} = d^n z d^n \bar{z} \]
\[ z = \frac{(x + i y)}{\sqrt{2}} \]
$\int d^{n}\chi d^{n}\bar{\chi} \exp(\chi^\dagger M \chi) = \det M$

Using shift invariance of integrals:

$\chi^\dagger \rightarrow \chi^\dagger - \eta^\dagger M^{-1} \eta$ \hspace{1cm} $\chi \rightarrow \chi - M^{-1} \eta$

We get:

$\int d^{n}\chi d^{n}\bar{\chi} \exp(\chi^\dagger M \chi + \eta^\dagger \chi + \chi^\dagger \eta) = (\det M) \exp(-\eta^\dagger M^{-1} \eta)$

Generalization for continuous spacetime argument and spin index $\Psi_\alpha(x)$

The determinant does not depend on fields or sources and can be absorbed into the overall normalization of the path integral.

$Z_0(\bar{\eta}, \eta) = \int D\Psi D\bar{\Psi} \exp\left[i \int d^4x \bar{\psi}(i\partial - m)\psi + \bar{\eta}\psi + \bar{\psi}\eta\right]$

$= \exp\left[i \int d^4x d^4y \bar{\eta}(x) \delta(x-y)\eta(y)\right]$.

For the Gaussian of $n$ complex variables we found:

In the continuum limit the “matrix” in our case is:

$M(x, y) = [-\partial_x^2 + m^2 - g\varphi(x)]\delta^4(x-y)$

We need to calculate the determinant of this “matrix”:

$M(x, y) = [-\partial_x^2 + m^2 - g\varphi(x)]\delta^4(x-y)$

We can write it as a product of two matrices:

$M(x, y) = \int d^4y M_0(x, y)\bar{M}(y, z)$

Where

$M_0(x, y) = (-\partial_x^2 + m^2)\delta^4(x-y)$,

$\bar{M}(y, z) = \delta^4(y-z) - g\Delta(y-z)\varphi(z)$.

Now we can calculate the determinant:

$\det M = \det M_0 \det \bar{M}$

$\bar{M} = I - G$

Independent of the background field (can be absorbed into the overall normalization)

Thus:

$Z(\varphi) = (\det \bar{M})^{-1}$

The path integral is given by:

$Z(\varphi) = (\det \bar{M})^{-1}$

Using $\det A = \exp \text{Tr} \ln A$ we get:

$\det \bar{M} = \exp \text{Tr} \ln \bar{M}$

$= \exp \text{Tr} \ln(I - G)$

$= \exp \text{Tr} \left[ -\sum_{n=1}^{\infty} \frac{1}{n} G^n \right]$}

Thus we find:

$Z(\varphi) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} G^n$

Where

$\text{Tr} G^n = g^n \int d^4x_1 \ldots d^4x_n \Delta(x_1-x_2)\varphi(x_2) \ldots \Delta(x_n-x_1)\varphi(x_1)$
we found:
\[ Z(\varphi) = \exp \frac{1}{n} \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} G^n \]
where
\[ \text{Tr} G^n = g^n \int d^4x_1 \ldots d^4x_n \Delta(x_1-x_2)\varphi(x_2) \ldots \Delta(x_n-x_1)\varphi(x_1) \]
It represents a connected diagram with \( n \) insertions of a vertex:
\[ \mathcal{L} = -\partial^\mu \chi^\dagger \partial_\mu \chi - m^2 \chi^\dagger \chi + g \varphi \chi^\dagger \chi \]
\[ S = n \text{ cyclic symmetry} \]
\[ \frac{1}{i} \Delta(x_1-x_2) \]
and the path integral is given by
\[ Z(\varphi) = \exp i\Gamma(\varphi) \quad i\Gamma(\varphi) = \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} G^n \]
\[ \text{sum of all connected diagrams} \]

Consider now a theory of a Dirac field:
\[ \mathcal{L} = i \bar{\Psi} \gamma^\mu \partial_\mu \Psi - m \bar{\Psi} \Psi + g \varphi \bar{\Psi} \Psi \]
we define the path integral:
\[ Z(\varphi) = \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \exp i \int d^4x \mathcal{L} \]
we fix \( Z(0) = 1 \)
for the gaussian of \( n \) complex Grassmann variables we found:
\[ \int d^4\psi d^4\psi \exp \left( -\frac{i}{2} \bar{\psi}_i M_{ij} \psi_j \right) \propto \det M \]
in the continuum limit the “matrix” in this case is:
\[ M_{\alpha\beta}(x, y) = [-i \partial_x + m - g \varphi(x)]_{\alpha\beta} \delta^4(x-y) \]
we need to calculate the determinant of this “matrix”
\[ \text{we can write it as a product of two matrices:} \]
\[ M = M_0 \tilde{M} \]
where
\[ M_{\alpha\beta}(x, y) = (-i \partial_\mu + m)_{\alpha\beta} \delta^4(x-y), \]
\[ \tilde{M}_{\alpha\beta}(y, z) = \delta_{\alpha\beta} \delta^4(y-z) - g S_{\beta\gamma}(y-z) \varphi(z) \]
\[ (-i \partial_\mu + m)_{\alpha\beta} S_{\beta\gamma}(y-z) = \delta_{\alpha\gamma} \delta^4(y-z) \]
now we can calculate the determinant:
\[ \det M = \det M_0 \det \tilde{M} \]
\[ \tilde{M} = I - G \]
\[ I_{\alpha\beta}(x, y) = \delta_{\alpha\beta} \delta^4(x-y) \]
\[ G_{\alpha\beta}(x, y) = g S_{\alpha\beta}(x-y) \varphi(y) \]
\[ \text{independent of the background field} \]
\[ \text{(can be absorbed into the overall normalization)} \]
\[ \text{thus:} \quad Z(\varphi) = \det \tilde{M} \]

the path integral is given by:
\[ Z(\varphi) = \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \exp i \int d^4x \mathcal{L} \]
using \( \det A = \exp \text{Tr} \ln A \) we get:
\[ \det \tilde{M} = \exp \text{Tr} \ln \tilde{M} \]
\[ = \exp \text{Tr} \ln (I - G) \]
\[ = \exp \text{Tr} \left[ -\sum_{n=1}^{\infty} \frac{1}{n} G^n \right] \]
\[ \text{thus we find:} \]
\[ Z(\varphi) = \exp -\sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} G^n \]
\[ \text{where} \]
\[ \text{Tr} G^n = g^n \int d^4x_1 \ldots d^4x_n \text{tr} S(x_1-x_2)\varphi(x_2) \ldots S(x_n-x_1)\varphi(x_1) \]
we found:

\[ Z(\varphi) = \exp - \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} G^n \]

where

\[ \text{Tr} G^n = g^n \int d^4x_1 \ldots d^4x_n \text{tr} S(x_1-x_2)\varphi(x_2) \ldots S(x_n-x_1)\varphi(x_1) \]

It represents a connected diagram with \( n \) insertions of a vertex:

\[ \mathcal{L} = i\bar{\psi}\gamma^\mu \psi - m\bar{\psi}\psi + g\varphi\bar{\psi}\psi \]

and the path integral is given by

\[ Z(\varphi) = \exp i\Gamma(\varphi) \]

\[ i\Gamma(\varphi) = \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} G^n \]

sum of all connected diagrams