Quantum electrodynamics (QED)

Quantum electrodynamics is a theory of photons interacting with the electrons and positrons of a Dirac field:

\[ \mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + i \bar{\Psi} \gamma^\mu \partial_\mu \Psi - m \bar{\Psi} \Psi + e \bar{\Psi} \gamma^\mu A_\mu \]

\[ e = -0.302822 \]

\[ \alpha = e^2/4\pi = 1/137.036 \]

Noether current of the lagrangian for a free Dirac field

\[ j^\mu(x) = e \bar{\Psi}(x) \gamma^\mu \Psi(x) \]

\[ \partial_\mu j^\mu(x) = \delta \mathcal{L}(x) - \frac{\delta S}{\delta \phi_a(x)} \delta \phi_a(x) \]

we want the current to be conserved and so we need to enlarge the gauge transformation also to the Dirac field:

\[ A^\mu(x) \rightarrow A^\mu(x) - \partial^\mu \Gamma(x) \]

\[ \Psi(x) \rightarrow \exp[-ie\Gamma(x)]\Psi(x) \]

\[ \bar{\Psi}(x) \rightarrow \exp[+ie\Gamma(x)]\bar{\Psi}(x) \]

global symmetry is promoted into local

symmetry of the lagrangian and so the current is conserved no matter if equations of motion are satisfied

We can write the QED lagrangian as:

\[ \mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + i \bar{\Psi} \gamma^\mu \partial_\mu \Psi - m \bar{\Psi} \Psi \]

\[ D_\mu \equiv \partial_\mu - ieA_\mu \]

covariant derivative

(the covariant derivative of a field transforms as the field itself)

\[ \Psi(x) \rightarrow \exp[-ie\Gamma(x)]\Psi(x) \]

\[ D_\mu \Psi(x) \rightarrow \exp[-ie\Gamma(x)]D_\mu \Psi(x) \]

and so the lagrangian is manifestly gauge invariant!

Proof:

\[ D_\mu \Psi \rightarrow \left( \partial_\mu - ie[A_\mu - \partial_\mu \Gamma] \right) \left( \exp[-ie\Gamma] \Psi \right) \]

\[ = \exp[-ie\Gamma] \left( \partial_\mu \Psi - ie(\partial_\mu \Gamma) \Psi - ie[A_\mu - \partial_\mu \Gamma] \Psi \right) \]

\[ = \exp[-ie\Gamma] \left( \partial_\mu - ieA_\mu \right) \Psi \]

\[ = \exp[-ie\Gamma] D_\mu \Psi . \]
We can also define the transformation rule for $D$:

$$D_\mu \rightarrow e^{-ie\Gamma} D_\mu e^{ie\Gamma}$$

then

$$D_\mu \Psi \rightarrow (e^{-ie\Gamma} D_\mu e^{ie\Gamma})(e^{-ie\Gamma} \Psi) = e^{-ie\Gamma} D_\mu \Psi,$$

as required.

Now we can express the field strength in terms of $D$'s:

$$D_\mu \equiv \partial_\mu - ieA_\mu$$

$$\left[D^\mu, D^\nu\right]\Psi(x) = -ieF^{\mu\nu}(x)\Psi(x)$$

$$F^{\mu\nu} = \frac{i}{e} \left[D^\mu, D^\nu\right]$$

Then we simply see:

$$F^{\mu\nu} \rightarrow \frac{i}{e} \left[ e^{-ie\Gamma} D^\mu e^{+ie\Gamma}, e^{-ie\Gamma} D^\nu e^{+ie\Gamma} \right]$$

$$= e^{-ie\Gamma} \left( \frac{i}{e} \left[D^\mu, D^\nu\right] \right) e^{+ie\Gamma}$$

$$= e^{-ie\Gamma} F^{\mu\nu} e^{+ie\Gamma}$$

$$= F^{\mu\nu}.$$
Nonabelian symmetries

Let's generalize the theory of two real scalar fields:  
\[ \mathcal{L} = -\frac{1}{2} \partial^\mu \varphi_1 \partial_\mu \varphi_1 - \frac{1}{2} \partial^\mu \varphi_2 \partial_\mu \varphi_2 - \frac{1}{2} m^2 (\varphi_1^2 + \varphi_2^2) - \frac{1}{16} \lambda (\varphi_1^2 + \varphi_2^2)^2 \]

to the case of \( N \) real scalar fields:

\[ \mathcal{L} = -\frac{1}{2} \partial^\mu \varphi_i \partial_\mu \varphi_i - \frac{1}{2} m^2 \varphi_i \varphi_i - \frac{1}{16} \lambda (\varphi_i \varphi_i)^2 \]

the lagrangian is clearly invariant under the \( \text{SO}(N) \) transformation:

\[ \varphi_i(x) \rightarrow R_{ij} \varphi_j(x) \]

orthogonal matrix with \( \det = 1 \)

\[ R^T = R^{-1} \]

\[ \det R = +1 \]

lagrangian has also the \( \mathbb{Z}_2 \) symmetry, \( \varphi_i(x) \rightarrow -\varphi_i(x) \), that enlarges \( \text{SO}(N) \) to \( \text{O}(N) \)

infinitesimal \( \text{SO}(N) \) transformation:

\[ R_{ij} = \delta_{ij} + \theta_{ij} + O(\theta^2) \]

real

\[ R_{ij}^{-1} = \delta_{ij} - \theta_{ij} \]

\[ \text{Im}(R^{-1}R)_{ij} = \text{Im} \sum_k R_{ki}R_{kj} = 0 \]

\( (N^2 \) linear combinations of \( \text{Im} \) parts \( = 0 \))

there are \( \frac{1}{2} N(N-1) \) linearly independent real antisymmetric matrices, and we can write:

\[ \theta_{jk} = -i \theta^a (T^a)_{jk} \]

hermitian, antisymmetric, \( N \times N \)

generator matrices of \( \text{SO}(N) \)

or \( R = e^{-i \theta^a T^a} \).

The commutator of two generators is a lin. comb. of generators:

\[ [T^a, T^b] = i f^{abc} T^c \]

we choose normalization: \( \text{Tr}(T^a T^b) = 2 \delta^{ab} \)

\[ f^{abc} = -\frac{1}{2} \text{Tr}([T^a, T^b] T^c) \]

structure constants of the \( \text{SO}(N) \) group
e.g. $\text{SO}(3)$:

\[
(T^a)_{ij} = -i\varepsilon^{aij}
\]

\[
[T^a, T^b] = i\varepsilon^{abc}T^c
\]

$\varepsilon^{123} = +1$

Levi-Civita symbol

consider now a theory of $N$ complex scalar fields:

\[
\mathcal{L} = -\partial^\mu \varphi_i^\dagger \partial_\mu \varphi_i - m^2 \varphi_i^\dagger \varphi_i - \frac{1}{4} \lambda (\varphi_i^\dagger \varphi_i)^2
\]

the lagrangian is clearly invariant under the $U(N)$ transformation:

\[
\varphi_i(x) \rightarrow U_{ij} \varphi_j(x)
\]

\[
U^\dagger = U^{-1}
\]

we can always write $U_{ij} = e^{-i\theta} \tilde{U}_{ij}$ so that $\text{det} \tilde{U} = +1$.

actually, the lagrangian has larger symmetry, $\text{SO}(2N)$:

\[
\varphi_j = (\varphi_{j1} + i\varphi_{j2})/\sqrt{2}
\]

\[
\varphi_j^\dagger \varphi_j = \frac{1}{2}(\varphi_{11}^2 + \varphi_{12}^2 + \ldots + \varphi_{N1}^2 + \varphi_{N2}^2)
\]

$\text{SU}(N)$ - group of special unitary $N\times N$ matrices

$U(N) = U(1) \times SU(N)$
infinitesimal \( SU(N) \) transformation:

\[
\tilde{U}_{ij} = \delta_{ij} - i \theta^a (T^a)_{ij} + O(\theta^2)
\]

or \( \tilde{U} = e^{-i \theta^a T^a} \).

there are \( N^2 - 1 \) linearly independent traceless hermitian matrices:

\[
\det \tilde{U} = +1
\]

\[
\ln \det A = \text{Tr} \ln A
\]

the structure coefficients are \( f_{abc} = 2 \varepsilon_{abc} \), the same as for \( SO(3) \)

**Nonabelian gauge theory**

Consider a theory of \( N \) scalar or spinor fields that is invariant under:

\[
\phi_i(x) \rightarrow U_{ij} \phi_j(x)
\]

for \( SU(N) \): a special unitary \( NxN \) matrix

for \( SO(N) \): a special orthogonal \( NxN \) matrix

In the case of \( U(1) \) we could promote the symmetry to local symmetry but we had to include a **gauge field** \( A_\mu(x) \) and promote ordinary derivative to **covariant derivative**:

\[
\phi(x) \rightarrow U(x) \phi(x) \quad U(x) = \exp[-ie \Gamma(x)] \quad D_\mu = \partial_\mu - ie A_\mu
\]

then the kinetic terms and mass terms: \( -\partial_\mu \phi \partial^\mu \phi \), \( m^2 \phi^\dagger \phi \), \( i \bar{\psi} D \psi \) and \( m \bar{\psi} \psi \), are gauge invariant. The transformation of covariant derivative in general implies that the gauge field transforms as:

\[
A_\mu(x) \rightarrow U(x) A_\mu(x) U^\dagger(x) + \frac{i}{e} U(x) \partial_\mu U^\dagger(x)
\]

for \( U(1) \): \( A_\mu(x) \rightarrow A_\mu(x) - \partial_\mu \Gamma(x) \)
Now we can easily generalize this construction for SU(N) or SO(N):

an infinitesimal SU(N) transformation:

\[ U_{jk}(x) = \delta_{jk} - ig\theta^a(x)(T^a)_{jk} + O(\theta^2) \]

generator matrices (hermitian and traceless):

\[ [T^a, T^b] = if^{abc}T^c \]

\[ \text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab} \]

structure constants (completely antisymmetric)

the SU(N) gauge field is a traceless hermitian NxN matrix transforming as:

\[ A_\mu(x) \rightarrow U(x)A_\mu(x)U^\dagger(x) + \frac{ig}{2} U(x)\partial_\mu U^\dagger(x) \]

\[ U(x) = \exp[-ig\Gamma^a(x)T^a] \]

the covariant derivative is:

\[ D_\mu = \partial_\mu - igA_\mu(x) \]

or acting on a field:

\[ (D_\mu \phi)_j(x) = \partial_\mu \phi_j(x) - igA_\mu(x)_{jk} \phi_k(x) \]

using covariant derivative we get a gauge invariant lagrangian

We define the field strength (kinetic term for the gauge field) as:

\[ F_{\mu\nu}(x) \equiv \frac{i}{g}[D_\mu, D_\nu] \]

\[ = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \]

it transforms as:

\[ F_{\mu\nu}(x) \rightarrow U(x)F_{\mu\nu}(x)U^\dagger(x) \]

\[ D_\mu \rightarrow U(x)D_\mu U^\dagger(x) \]

not gauge invariant separately

and so the gauge invariant kinetic term can be written as:

\[ \mathcal{L}_{\text{kin}} = -\frac{1}{2} \text{Tr}(F^{\mu\nu}F_{\mu\nu}) \]
we can expand the gauge field in terms of the generator matrices:

\[ A_\mu(x) = A_\mu^a(x)T^a \]

that can be inverted:

\[ A_\mu^a(x) = 2 \text{Tr} A_\mu(x)T^a \quad \text{Tr}(T^aT^b) = \frac{1}{2} \delta^{ab} \]

similarly:

\[ F_{\mu\nu}^a(x) = F_{\mu\nu}^a T^a, \quad F_{\mu\nu}^a(x) = 2 \text{Tr} F_{\mu\nu} T^a. \]

\[
F_{\mu\nu}(x) = \frac{i}{2}[D_\mu, D_\nu]
\]

\[
= \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \rightarrow F_{\mu\nu}^c T^c = (\partial_\mu A_\nu^c - \partial_\nu A_\mu^c)T^c - igA_\mu^a A_\nu^b [T^a, T^b]
\]

\[
[T^a, T^b] = if^{abc}T^c
\]

thus we have:

\[ F_{\mu\nu}^c = \partial_\mu A_\nu^c - \partial_\nu A_\mu^c + gf^{abc}A_\mu^a A_\nu^b \]

\[
\mathcal{L}_{\text{kin}} = -\frac{1}{2} \text{Tr}(F^{\mu\nu}F_{\mu\nu})
\]

the kinetic term can be also written as:

\[ F_{\mu\nu}(x) = F_{\mu\nu}^a T^a \quad \text{Tr}(T^aT^b) = \frac{1}{2} \delta^{ab} \]

\[ F_{\mu\nu}^c = \partial_\mu A_\nu^c - \partial_\nu A_\mu^c + gf^{abc}A_\mu^a A_\nu^b \]

Example, quantum chromodynamics - QCD:

\[ \mathcal{L} = i\overline{\Psi}_i D_\mu^i \Psi^I_I - m_I \overline{\Psi}_I \Psi_I - \frac{1}{2} \text{Tr}(F^{\mu\nu}F_{\mu\nu}) \]

flavor index: up, down, strange, charm, top, bottom

\[ (D_\mu)_{ij} = \delta_{ij} \partial_\mu - igA_\mu^a T^a_{ij} \]

1, ..., 8 gluons (massless spin 1 particles)

color index: 1, 2, 3

in general, scalar and spinor fields can be in different representations of the group, \( T^a_R \); gauge invariance requires that the gauge fields transform independently of the representation.
Group representations

A representation of a group is specified by a set of hermitian matrices that obey:

\[ [T^a_R, T^b_R] = i f^{abc} T^c_R \]

(the original set of \( N \times N \) dimensional matrices for \( SU(N) \) or \( SO(N) \) corresponds to the fundamental representation)

Taking the complex conjugate we see that \( -(T^a_R)^* \) is also a representation!

- R is real if \( -(T^a_R)^* = T^a_R \)
- or if there is a unitary transformation \( T^a_R \rightarrow U^{-1} T^a_R U \) that makes \( -(T^a_R)^* = T^a_R \)

- R is pseudoreal if it is not real but there is a transformation such that \( -(T^a_R)^* = V^{-1} T^a_R V \)
  - e.g. fundamental rep of \( SU(2) \)
  - \( -(\frac{1}{2} \sigma^a)^* = V^{-1}(\frac{1}{2} \sigma^a)V, \quad V = \sigma_2 \)

- If R is not real or pseudoreal then it is complex

The adjoint representation \( A \):

\[ (T^a_A)_{bc} = -i f^{abc} \]

- A is a real representation
- \( -(T^a_A)^* = T^a_A \)

The dimension of the adjoint representation, \( D(A) = \# \) of generators

= the dimension of the group

to see that \( T^a_A \) satisfy commutation relations we use the Jacobi identity:

\[ f^{abc} f^{def} + f^{bde} f^{fac} + f^{def} f^{abc} = 0 \]

follows from:

\[ \text{Tr} T^a ([T^a, T^b], T^c) + ([T^b, T^c], T^a) + ([T^c, T^a], T^b) = 0 \]

\( [T^a_R, T^b_R] = i f^{abc} T^c_R \)

\[ [T^a_A, T^c_A] = i f^{acd} T^d_A \]
The index of a representation $T(R)$:

$$\text{Tr}(T^a_R T^b_R) = T(R)\delta^{ab}$$

The quadratic Casimir $C(R)$:

$$T^a_R T^a_R = C(R)$$

multiplies the identity matrix, commutes with every generator, homework S-69.2

Useful relation:

$$T(R)D(A) = C(R)D(R)$$

For $SU(N)$:

- $T(N) = \frac{1}{2}$
- $T(A) = N$
- $D(A) = N^2 - 1$

For $SO(N)$:

- $T(N) = 2$
- $T(A) = 2N - 4$
- $D(A) = \frac{1}{2}N(N-1)$

A representation is reducible if there is a unitary transformation

$$T^a_R \rightarrow U^{-1} T^a_R U$$

that brings all the generators to the same block diagonal form (with at least two blocks); otherwise it is irreducible.

For example, consider a reducible representation $R$ that can put into two blocks, then $R$ is a direct sum representation:

$$R = R_1 \oplus R_2$$

and we have:

$$D(R_1 \oplus R_2) = D(R_1) + D(R_2)$$

$$\text{Tr}(T^a_R T^b_R) = T(R)\delta^{ab}$$

$$T(R_1 \oplus R_2) = T(R_1) + T(R_2)$$
Consider a field that carries two group indices \( \varphi_{ij}(x) \):

\[
\begin{array}{c}
R_1 \\
R_2
\end{array}
\]

then the field is in the direct product representation:

\[
R_1 \otimes R_2
\]

The corresponding generator matrix is:

\[
(T^a_{R_1 \otimes R_2})_{ij,kl} = (T^a_{R_1})_{ij} \delta_{kl} + \delta_{ij} (T^a_{R_2})_{kl}
\]

and we have:

\[
D(R_1 \otimes R_2) = D(R_1)D(R_2)
\]

\[
T(R_1 \otimes R_2) = T(R_1)D(R_2) + D(R_1)T(R_2)
\]

to prove this we use the fact that \((T^a_R)_{ii} = 0\).

We will use the following notation for indices of a complex representation:

\[
\varphi_i \quad i = 1, 2, \ldots, D(R)
\]

Hermitian conjugation changes \(R\) to \(\overline{R}\) and for a field in the conjugate representation we will use the upper index

\[
(\varphi_i)^\dagger = \varphi^\dagger i
\]

we write generators as:

\[
(T^a_R)_{ij}
\]

indices are contracted only if one is up and one is down!

An infinitesimal group transformation of \(\varphi_i\) is:

\[
\varphi_i \rightarrow (1 - i \theta^a T^a_R)_{ij} \varphi_j
\]

\[
= \varphi_i - i \theta^a (T^a_R)_{ij} \varphi_j
\]
generator matrices for $\overline{R}$ are then given by

$$(T^a_R)_{ij} = -(T^a_R)_{ji} \quad \quad T^a_R = -(T^a_R)^*$$

we trade complex conjugation for transposition

and an infinitesimal group transformation of $\varphi^+^i$ is:

$$\varphi^+^i \rightarrow (1 - i\theta^a T^a_R)_{ij} \varphi^+^j$$

$$= \varphi^+^i - i\theta^a (T^a_R)_{ij} \varphi^+^j$$

$$(T^a_R)_{ij} = -(T^a_R)_{ji}$$

$$= \varphi^+^i + i\theta^a (T^a_R)_{ij} \varphi^+^j$$

$$\varphi_i \rightarrow (1 - i\theta^a T^a_R)_{ij} \varphi_j$$

$$= \varphi_i - i\theta^a (T^a_R)_{ij} \varphi_j$$

is invariant!

Consider the Kronecker delta symbol

$$\delta^k_l \rightarrow (1 + i\theta^a T^a_R)_{ik} \delta^k_l (1 + i\theta^a T^a_R)^i_l$$

$$= (1 + i\theta^a T^a_R)_{ik} \delta^k_l (1 - i\theta^a T^a_R)^i_l$$

$$= \delta^k_l + O(\theta^2).$$

is an invariant symbol of the group!

this means that the product of the representations $R$ and $\overline{R}$ must contain the singlet representation $1$, specified by $T^a_1 = 0$.

Thus we can write:

$$R \otimes \overline{R} = 1 \oplus \ldots$$
Another invariant symbol:

\[
(T^b_R)_{ij} \rightarrow (1 - i\theta^a T^a_R)_{ik}(1 - i\theta^a T^a_R)^j_l(1 - i\theta^a T^a_A)^{bc}(T^c_R)_{kl}
\]

\[
R = (T^b_R)_{ij} - i\theta^a[(T^a_R)_{ik}(T^b_R)^{j_l} + (T^a_R)^{jl}(T^b_R)_{ij} + (T^a_A)^{bc}(T^c_R)_{ij}]
\]

\[+ O(\theta^2).\]

\[
(T^a_R)_{ij} = -(T^a_R)^{ij} \quad (T^a_A)^{bc} = -if^{abc}
\]

\[
[\ldots] = (T^a_R)_{ik}(T^b_R)^{j_l} - (T^a_R)^{jl}(T^b_R)_{ij} - if^{abc}(T^c_R)_{ij}
\]

\[
= (T^a_R - T^b_R)_{ij} - (T^b_R T^a_R)_{ij} - if^{abc}(T^c_R)_{ij}
\]

\[= 0 ,
\]

this implies that:

\[
R \otimes \overline{R} \otimes A = 1 \oplus \ldots
\]

must contain the singlet representation!

\[
R \otimes \overline{R} \otimes A = 1 \oplus \ldots
\]

multiplying by \(A\) we find:

\[
R \otimes \overline{R} = A \oplus \ldots
\]

(A is real)

combining it with a previous result we get

\[
R \otimes \overline{R} = 1 \oplus A \oplus \ldots
\]

the product of a representation with its complex conjugate is always reducible into a sum that contains at least the singlet and the adjoint representations!

For the fundamental representation \(N\) of \(SU(N)\) we have:

\[
N \otimes \overline{N} = 1 \oplus A
\]

\[
D(1) = 1
\]

\[
D(N) = D(\overline{N}) = N
\]

\[
D(A) = N^2 - 1
\]

(no room for anything else)
Consider a real representation $R$:  
\[ \mathbf{R} \otimes \mathbf{R} = 1 \oplus \mathbf{A} \oplus \ldots \]
implies the existence of an invariant symbol with two $R$ indices

\[ \delta_{ij} \to (1 - i\theta^a T^a_{R})^i_k (1 - i\theta^a T^a_{R})^j_l \delta_{kl} \]
\[ = \delta_{ij} - i\theta^a [(T^a_{R})_{ij} \pm (T^a_{R})_{ji}] + O(\theta^2) \]

For the fundamental representation $\mathbf{N}$ of $\text{SO}(N)$ we have:

\[ \mathbf{N} \otimes \mathbf{N} = 1_s \oplus \mathbf{A}_A \oplus \mathbf{S}_S \]

\[ D(1) = 1 \]
\[ D(\mathbf{N}) = N \]
\[ D(\mathbf{A}) = \frac{1}{2} N(N-1) \]
\[ D(\mathbf{S}) = \frac{1}{2} N(N+1)-1 \]
\[ \phi_{ij} = \phi_{ji} \]

$R$ is pseudoreal if it is not real but there is a transformation such that $-(T^a_{R})^* = V^{-1}T^a_{R}V$

Consider now a pseudoreal representation:

\[ \mathbf{R} \otimes \mathbf{R} = 1 \oplus \mathbf{A} \oplus \ldots \]
still holds but the Kronecker delta is not the corresponding invariant symbol:

\[ \delta_{ij} \to (1 - i\theta^a T^a_{R})^i_k (1 - i\theta^a T^a_{R})^j_l \delta_{kl} \]
\[ = \delta_{ij} - i\theta^a [(T^a_{R})_{ij} \pm (T^a_{R})_{ji}] + O(\theta^2) \]
the only alternative is to have the singlet appear in the antisymmetric part of the product. For $\text{SU}(N)$ another invariant symbol is the Levi-Civita symbol with $\mathbf{N}$ indices:

\[ \varepsilon_{i_1 \ldots i_N} \to U_{i_1 j_1} \ldots U_{i_N j_N} \varepsilon_{j_1 \ldots j_N} \]
\[ = (\det U)\varepsilon_{i_1 \ldots i_N} . \]
similarly for $\varepsilon^{i_1 \ldots i_N}$.

For $\text{SU}(2)$:

\[ 2 \otimes 2 = 1_A \oplus 3_S \]
we can use $\varepsilon^{ij}$ and $\varepsilon_{ij}$ to raise and lower $\text{SU}(2)$ indices; if $\phi^i$ is in the $2$ representation, then we can get a field in the $\bar{2}$ representation by raising the index: $\phi^i = \varepsilon^{ij} \phi_j$. 

Another invariant symbol of interest is \( f^{abc} \):

\[
(T^a_L)_{bc} = -if^{abc}
\]

generator matrices in any rep. are invariant, or

\[
T(R)f^{abc} = -i\text{ Tr}(T^a_R[T^b_R, T^c_R])
\]

the right-hand side is obviously invariant.

Very important invariant symbol is the anomaly coefficient of the rep.:

\[
A(R)d^{abc} = \frac{1}{2}\text{ Tr}(T^a_R \{ T^b_R, T^c_R \})
\]

is completely symmetric

normalized so that \( A(N) = 1 \) for \( SU(N) \) with \( N \geq 3 \).

Since \( (T^a_R)^i_j = -(T^a_R)^j_i \) we have:

\[
A(R) = -A(R)
\]

for real or pseudoreal representations \( A(R) = 0 \).

e.g. for \( SU(2) \), all representation are real or pseudoreal and \( A(R) = 0 \) for all of them

we also have:

\[
A(R_1 \oplus R_2) = A(R_1) + A(R_2),
\]

\[
A(R_1 \otimes R_2) = A(R_1)D(R_2) + D(R_1)A(R_2)
\]

The path integral for photons

We will discuss the path integral for photons and the photon propagator more carefully using the Lorentz gauge:

\[
Z_0(J) = \int DA \ e^{iS_0}, \quad \mathcal{L} = \frac{i}{2}A_\mu (g^{\mu\nu}\partial^2 - \partial^\mu \partial^\nu)A_\nu + J^\mu A_\mu
\]

as in the case of scalar field we Fourier-transform to the momentum space:

\[
S_0 = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \left[ -\vec{\varphi}(k)(\vec{k}^2 + m^2)\varphi(-k) + \vec{J}(k)\varphi(-k) + \vec{\varphi}(-k)\varphi(k) \right]
\]

\[
S_0 = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \left[ -\vec{A}_\mu(k)(\vec{k}^2 g^{\mu\nu} - k^\mu k^\nu)\vec{A}_\nu(-k) + \vec{J}_\mu(k)\vec{A}_\mu(-k) + \vec{\varphi}(-k)\vec{A}_\mu(k) \right]
\]

we shift integration variables so that mixed terms disappear...

\[
\tilde{\chi}(k) = \varphi(k) - \frac{\vec{J}(k)}{k^2 + m^2}
\]

Problem: the matrix has zero eigenvalue and cannot be inverted.
To see this, note:

$$k^2 g^{\mu\nu} - k^\mu k^\nu = k^2 P^{\mu\nu}(k)$$

where

$$P^{\mu\nu}(k) \equiv g^{\mu\nu} - k^\mu k^\nu / k^2$$

is a projection matrix

$$P^{\mu\nu}(k) P^{\nu\lambda}(k) = P^{\mu\lambda}(k)$$

and so the only allowed eigenvalues are 0 and +1

Since

$$P^{\mu\nu}(k) k_\nu = 0$$

$$g_{\mu\nu} P^{\mu\nu}(k) = 3$$

it has one 0 and three +1 eigenvalues.

We can decompose the gauge field \( \tilde{A}_\mu(k) \) into components aligned along a set of linearly independent four-vectors, one of which is \( k_\mu \) and then this component does not contribute to the quadratic term because

$$P^{\mu\nu}(k) k_\nu = 0$$

and it doesn’t even contribute to the linear term because

$$\partial^\mu J_\mu(x) = 0 \quad \rightarrow \quad k_\mu \tilde{J}_\mu(k) = 0$$

and so there is no reason to integrate over it; we define the path integral as integral over the remaining three basis vector; these are given by

$$k^\mu \tilde{A}_\mu(k) = 0$$

which is equivalent to

$$\partial^\mu A_\mu(x) = 0$$

Lorentz gauge
Within the subspace orthogonal to $k_\mu$, the projection matrix is simply the identity matrix and the inverse is straightforward; thus we get:

$$Z_0(J) = \exp \left[ \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \tilde{J}_\mu(k) \frac{P^{\mu\nu}(k)}{k^2 - i\epsilon} \tilde{J}_\nu(-k) \right]$$

$$= \exp \left[ \frac{i}{2} \int d^4x \, d^4y \, J_\mu(x) \Delta^{\mu\nu}(x-y) J_\nu(y) \right]$$

going back to the position space

$$\Delta^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} \, e^{i(k-x)(x-y)} \frac{P^{\mu\nu}(k)}{k^2 - i\epsilon}$$

propagator in the Lorentz gauge (Landau gauge)

we can again neglect the term with momenta because the current is conserved and we obtain the propagator in the Feynman gauge:

$$\Delta^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} \, e^{i(k-x)(x-y)} \tilde{\Delta}^{\mu\nu}(k)$$

The path integral for nonabelian gauge theory

Now we want to evaluate the path integral for nonabelian gauge theory:

$$Z(J) \propto \int \mathcal{D}A \, e^{iS_{YM}(A,J)}$$

$$S_{YM}(A,J) = \int d^4x \left[ -\frac{1}{4} F^{a\mu\nu} F_{a\mu\nu}^a + J^{a\mu} A_{\mu}^a \right]$$

for $U(1)$ gauge theory, the component of the gauge field parallel to the four-momentum $k^\mu$ did not appear in the action and so it should not be integrated over; since the $U(1)$ gauge transformation is of the form $A_\mu(x) \rightarrow A_\mu(x) - \partial_\mu \Gamma(x)$, excluding the components parallel to $k^\mu$ removes the gauge redundancy in the path integral.

nonabelian gauge transformation is nonlinear:

$$A_\mu(x) \rightarrow U(x) A_\mu(x) U^\dagger(x) + \frac{i}{g} U(x) \partial_\mu U^\dagger(x)$$

$$A_\mu(x) = A_\mu^a(x) T^a$$
for an infinitesimal transformation:
\[ U(x) = I - ig\theta(x) + O(\theta^2) \]
\[ = I - ig\theta^a(x)T^a + O(\theta^2) \]

we have:
\[ A_\mu(x) \rightarrow U(x)A_\mu(x)U^\dagger(x) + \frac{i}{g}U(x)\partial_\mu U^\dagger(x) \]
\[ A_\mu(x) \rightarrow A_\mu(x) + ig[A_\mu(x), \theta(x)] - \partial_\mu \theta(x) \]

or, in components:
\[ A_\mu^a(x) \rightarrow A_\mu^a(x) - gf^{abc}A_\mu^b(x)\theta^c(x) - \partial_\mu \theta^a(x) \]
\[ = A_\mu^a(x) - [\delta^{ac}\partial_\mu + gf^{abc}A_\mu^b(x)]\theta^c(x) \]
\[ = A_\mu^a(x) - [\delta^{ac}\partial_\mu - igA_\mu^b(-if^{bac})]\theta^c(x) \]
\[ = A_\mu^a(x) - D_\mu^a\theta^c(x) , \]

the covariant derivative in the adjoint representation
(instead of \( \partial_\mu \) that we have for the U(1) transformation)

we have to remove the gauge redundancy in a different way!

Consider an ordinary integral of the form:
\[ Z \propto \int dx \, dy \, e^{iS(x)} \]

the integral over \( y \) is redundant
we can simply drop it and define:
\[ Z \equiv \int dx \, e^{iS(x)} \]

this is how we dealt with gauge redundancy in the abelian case

or we can get the same result by inserting a delta function:
\[ Z = \int dx \, dy \, \delta(y) \, e^{iS(x)} \]

the argument of the delta function can be shifted by an arbitrary function of \( x \)
this is what we are going to do for the nonabelian case
\[ Z = \int dx \, dy \, \delta(y - f(x)) \, e^{iS(x)} \]

if \( y = f(x) \) is a unique solution of \( G(x, y) = 0 \) for fixed \( x \), we can write:

\[ \delta(G(x, y)) = \frac{\delta(y - f(x))}{|\partial G/\partial y|} \]

then we have:

\[ Z = \int dx \, dy \, \frac{\partial G}{\partial y} \delta(G) \, e^{iS} \]

we dropped the abs. value

generalizing the result to an integral over \( n \) variables:

\[ Z = \int d^n x \, d^n y \, \det \left( \frac{\partial G_i}{\partial y_j} \right) \prod_i \delta(G_i) \, e^{iS} \]

Now we translate this result to path integral over nonabelian gauge fields:

\[ Z = \int d^n x \, d^n y \, \det \left( \frac{\partial G_i}{\partial y_j} \right) \prod_i \delta(G_i) \, e^{iS} \]

\( i \) index now represents \( x \) and \( a \)

\( x \) and \( y \) \( \rightarrow \) \( A_\mu^a(x) \) \( \theta^a(x) \)

\( y \) \( \rightarrow \) \( \theta^a(x) \) \( G \) becomes the gauge fixing function:

\( \text{for} \ R_\xi \ \text{gauge we use:} \)

\[ G^a(x) \equiv \partial^\mu A_\mu^a(x) - \omega^a(x) \]

\( \text{fixed, arbitrarily chosen function of} \ x \)

\[ Z(J) \propto \int \mathcal{D}A \ \det \left( \frac{\delta G}{\delta \theta} \right) \prod_{x,a} \delta(G) \, e^{iS_{\text{YM}}} \]

\[ S_{\text{YM}}(A, J) = \int d^4 x \left[ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}^a + J_\mu A^a_\mu \right] \]
let’s evaluate the functional derivative:

\[ G^a(x) \equiv \partial^\mu A^a_\mu(x) - \omega^a(x) \]

\[ A^a_\mu(x) \rightarrow A^a_\mu(x) - D_\mu^{ac} \theta^c(x) \]

\[ G^a(x) \rightarrow G^a(x) - \partial^\mu D^{ab}_\mu \theta^b(x) \]

and we find:

\[ \frac{\delta G^a(x)}{\delta \theta^b(y)} = -\partial^\mu D^{ab}_\mu \delta^4(x-y) \]

Recall, the functional determinant can be written as a path integral over complex Grassmann variables:

\[ \int d^4 \tilde{\psi} \, d^4 \psi \exp(-i \tilde{\psi}_i M_{ij} \psi_j) \propto \det M \]

\[ \det \frac{\delta G^a(x)}{\delta \theta^b(y)} \propto \int Dc \, D\bar{c} \, e^{i S_{gh}} \]

\[ S_{gh} = \int d^4x \, L_{gh} \]

where:

\[ L_{gh} = \bar{c}^a \partial^\mu D^{ab}_\mu \, c^b \]

Faddeev-Popov ghosts

the ghost lagrangian can be further written as:

\[ L_{gh} = \bar{c}^a \partial^\mu D^{ab}_\mu \, c^b \]

we drop the total divergence

\[ D^{ac}_\mu = \delta^{ac} \partial_\mu - ig A^{b\mu}_\mu (T^b_A)^{ac} \]

\[ = -\partial^\mu \bar{c}^a D^{ab}_\mu \, c^b \]

\[ = -\partial^\mu \bar{c}^a \partial_\mu c^a + ig \partial^\mu \bar{c}^a A^c_\mu (T^b_A)^{ac} \, c^b \]

Comments:

- ghost fields interact with the gauge field; however ghosts do not exist and we will see later (when we discuss the BRST symmetry) that the amplitude to produce them in any scattering process is zero. The only place they appear is in loops! Since they are Grassmann fields, a closed loop of ghost lines in a Feynman diagram comes with a minus sign!

- For abelian gauge theory \( f^{abc} = 0 \) and thus there is no interaction term for ghost fields; we can absorb its path integral into overall normalization.
At this point we have:

\[
Z(J) \propto \int D\mathcal{A} \det\left(\frac{\delta G}{\delta \theta}\right) \prod_{x,a} \delta(G) \, e^{iS_{\text{YM}}}
\]

\[
\det \frac{\delta G^a(x)}{\delta \theta^b(y)} \propto \int D\mathcal{C} D\overline{\mathcal{C}} \, e^{iS_{\text{gh}}}
\]

\[
G^a(x) \equiv \partial^\mu A^a_\mu(x) - \omega^a(x)
\]

fixed, arbitrarily chosen function of \( x \)

The path integral is independent of \( \omega^a(x) \)! Thus we can multiply it by arbitrary functional of \( \omega \) and perform a path integral over \( \omega \); the result changes only the overall normalization of \( Z(J) \).

we can multiply \( Z(J) \) by:

\[
\exp \left[ -\frac{i}{2\xi} \int d^4x \, \omega \cdot \omega \right]
\]

integral over \( \omega \) is trivial

\[
\mathcal{L}_{\text{gf}} = -\frac{1}{2} \xi^{-1} \partial^\mu A^a_\mu \partial^\nu A^a_\nu
\]

gauge fixing term

our final result is:

\[
Z(J) \propto \int D\mathcal{A} D\overline{\mathcal{C}} D\mathcal{C} \exp\left(iS_{\text{YM}} + iS_{\text{gh}} + iS_{\text{gf}}\right)
\]

next time we will derive Feynman rules from this action...

---

The Feynman rules for nonabelian gauge theory

based on S-72

The lagrangian for nonabelian gauge theory is:

\[
\mathcal{L}_{\text{YM}} = -\frac{1}{4} F^{\mu\nu}_{\mathcal{A}} F^{\mathcal{A}}_{\mu\nu}
\]

\[
= -\frac{1}{4} \left( \partial^\mu A^{e\mu} - \partial^\nu A^{e\mu} + g f^{abe} A^{a\mu} A^{b\nu} \right) \left( \partial_\mu A^{e}_\nu - \partial_\nu A^{e}_\mu + g f^{cde} A^{c}_\mu A^{d}_\nu \right)
\]

\[
= -\frac{1}{2} \partial^\mu A^{e\mu} \partial_\mu A^{e}_\nu + \frac{1}{2} \partial^\mu A^{e\mu} \partial_\nu A^{e}_\mu
\]

\[
- g f^{abe} A^{a\mu} A^{b\nu} \partial_\mu A^{e}_\nu - \frac{1}{4} g^2 f^{abc} f^{cde} A^{a\mu} A^{b\nu} A^{c}_\mu A^{d}_\nu.
\]

the gauge fixing term for \( R_\xi \) gauge:

\[
\mathcal{L}_{\text{gf}} = -\frac{1}{2} \xi^{-1} \partial^\mu A^{e}_\mu \partial^\nu A^{e}_\nu
\]

we can write the gauge fixed lagrangian in the form:

\[
\mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{gf}} = \frac{1}{2} A^{e\mu} (g_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu) A^{e\nu} + \frac{1}{2} \xi^{-1} A^{e\mu} \partial_\mu \partial_\nu A^{e\nu}
\]

\[
- g f^{abe} A^{a\mu} A^{b\nu} \partial_\mu A^{e}_\nu - \frac{1}{4} g^2 f^{abc} f^{cde} A^{a\mu} A^{b\nu} A^{c}_\mu A^{d}_\nu
\]
The gluon propagator in the $R_\xi$ gauge:

$$\mathcal{L}_{YM} + \mathcal{L}_{gf} = \frac{1}{2} A^{e\mu}(g_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu) A^{e\nu} + \frac{1}{2} \xi^{-1} A^{e\mu} \partial_\mu \partial_\nu A^{e\nu}$$

$$- g f^{abc} A^{a\mu} A^{b\nu} \partial_\mu A^{c\nu} - \frac{1}{4} g^2 f^{abc} f^{cde} A^{a\mu} A^{b\nu} A^{c\rho} A^{d\sigma}$$

Going to the momentum space and taking the inverse of the quadratic term:

$$\tilde{\Delta}_{\mu\nu}^{ab}(k) = \frac{\delta^{ab}}{k^2 - i\epsilon} \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} + \xi \frac{k_\mu k_\nu}{k^2} \right)$$

The three-gluon vertex:

$$\mathcal{L}_{YM} + \mathcal{L}_{gf} = \frac{1}{2} A^{e\mu}(g_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu) A^{e\nu} + \frac{1}{2} \xi^{-1} A^{e\mu} \partial_\mu \partial_\nu A^{e\nu}$$

$$- g f^{abc} A^{a\mu} A^{b\nu} \partial_\mu A^{c\nu} - \frac{1}{4} g^2 f^{abc} f^{cde} A^{a\mu} A^{b\nu} A^{c\rho} A^{d\sigma}$$

All photons outgoing. The derivative acting on an outgoing particle brings (-i momentum) of the particle.

$$i V_{\mu\nu\rho}^{abc}(p, q, r) = i(-g f^{abc})(-ir_\mu g_{\nu\rho})$$

$$+ \left[ 5 \text{ permutations of } (a, \mu, p), (b, \nu, q), (c, \rho, r) \right]$$

$$= g f^{abc}[(q-r)_\mu g_{\nu\rho} + (r-p)_\nu g_{\rho\mu} + (p-q)_\rho g_{\mu\nu}] .$$
The four-gluon vertex:

\[
\mathcal{L}_{YM} + \mathcal{L}_{gf} = \frac{1}{2} A^{\mu}(g_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu) A^{\nu} + \frac{1}{2} \xi^{-1} A^{\mu} \partial_\mu \partial_\nu A^{\nu} - g f^{abc} A^{\mu} A^{\nu} \partial_\mu A_c^{\nu} - \frac{1}{4} g^2 f^{abcde} A^{\mu} A^{\nu} A^{\rho} A^{\sigma}
\]

\[i \mathcal{V}_{\mu\nu\rho\sigma}^{abcd} = -i g^2 f^{abc} f^{cde} g_{\mu\rho} g_{\nu\sigma} + [5 \text{ permutations of } (b,\nu), (c,\rho), (d,\sigma)]
\]

\[= -i g^2 \left[ f^{abc} f^{cde} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) + f^{ace} f^{dbe} (g_{\mu\sigma} g_{\rho\nu} - g_{\mu\nu} g_{\rho\sigma}) + f^{ade} f^{bce} (g_{\mu\nu} g_{\sigma\rho} - g_{\mu\rho} g_{\sigma\nu}) \right].\]

The ghost lagrangian: we need ghosts for loop calculations

\[
\mathcal{L}_{gh} = -\partial^\mu c^b D^{\mu c}_{\mu} c^c
\]

\[= -\partial^\mu c^c \partial_\mu c^c + ig \partial^\mu c^b A^a_\mu (T^a_A)^{bc} c^c
\]

\[= -\partial^\mu c^c \partial_\mu c^c + gf^{abc} A^a_\mu \partial^\mu c^b c^c.
\]

The ghost propagator:

\[
\tilde{\Delta}^{ab}(k^2) = \frac{\delta^{ab}}{k^2 - i\epsilon}
\]
The ghost-ghost-gluon vertex:

\[ \mathcal{L}_{gh} = -\partial^\mu \bar{c}^c \partial_\mu c^c + g f^{abc} A^a_\mu \partial^\mu \bar{c}^b c^c \]

the derivative acting on an outgoing particle brings \((-i\text{ momentum})\) of the particle

ghosts are complex scalars so their propagator carry a charge arrow

\[ i V_{\mu}^{abc}(q, r) = i (g f^{abc})(-iq_\mu) = g f^{abc} q_\mu . \]

Finally we can include quarks:

\[ \mathcal{L}_q = i \bar{\Psi}_i \gamma^\mu i\sigma_{ij} \Psi_j - m \bar{\Psi}_i \Psi_i = i \bar{\Psi}_i \gamma^\mu \psi_j - m \bar{\Psi}_i \psi_i + g A^a_\mu \bar{\Psi}_i \gamma^\mu T^a_{ij} \Psi_j \]

propagator:

\[ \tilde{S}_{ij}(p) = \frac{(-\not{p} + m) \delta_{ij}}{p^2 + m^2 - i\epsilon} \]

vertex:

\[ i V_{ij}^{\mu a} = ig \gamma^\mu T^a_{ij} \]

for fields in different representations we would have \((T^a_R)_{ij}\).
The beta function in nonabelian gauge theory

based on S-73

The complete (renormalized) lagrangian for nonabelian gauge theory is:

\[
\mathcal{L} = \frac{1}{2} Z_3 A^{a \mu} (g_{\mu \nu} \partial^2 - \partial_\mu \partial_\nu) A^{a \nu} + \frac{1}{2} \xi^{-1} A^{a \mu} \partial_\mu \partial_\nu A^{a \nu} - Z_3 g f^{abc} A^{a \mu} A^{b \nu} \partial_\mu A^{c \nu} - \frac{1}{4} Z_4 g^2 f^{abc} f^{def} A^{a \mu} A^{b \nu} A^{c \rho} A^{d \sigma} \\
- Z_2 \partial^\mu \tilde{C}^a \partial_\mu C^a + Z_1' g f^{abc} A^{a \mu} \partial_\mu \tilde{C}^a C^b \\
+ i Z_2 \bar{\psi}_i \partial \psi_i - Z_m m \bar{\psi}_i \psi_i + Z_1 g A^{a \mu} \bar{\psi}_i \gamma^\mu T^{a \mu}_i \psi_j.
\]

from gauge invariance we expect:

\[
g_0^2 = \frac{Z_2^2 Z_3}{Z_2^2 Z_3} g^2 \tilde{\mu}^\varepsilon = \frac{Z_1'}{Z_3^2 Z_3} g^2 \tilde{\mu}^\varepsilon = \frac{Z_3^2 g^2 \tilde{\mu}^\varepsilon}{Z_3^2} = \frac{Z_4 g^2 \tilde{\mu}^\varepsilon}{Z_3^2}
\]

Slavnov-Taylor identities
(non-abelian analogs of Ward identities)

There is only one diagram contributing at one loop level:

\[
i \Sigma(p) = (i Z_1 e)^2 \left( \frac{1}{i} \right)^2 \int \frac{d^4 \ell}{(2\pi)^4} \left[ \gamma^\nu \tilde{S}(\psi' + \ell) \gamma^\mu \right] \tilde{\Delta}_{\mu \nu}(\ell) \\
- i(Z_2 - 1)\psi - i(Z_m - 1)m + O(e^4). \]

the photon propagator
in the Feynman gauge:

\[
\tilde{\Delta}_{\mu \nu}(\ell) = \frac{g_{\mu \nu}}{\ell^2 + m_\gamma^2 - i\epsilon}
\]

fictitious photon mass
following the usual procedure:

\[ i\Sigma(\phi) = e^2 \mu^e \int_0^1 dx \int \frac{d^d q}{(2\pi)^d} \frac{N}{(q^2 + D)^2} \]

\[ - i(Z_2 - 1)\phi - i(Z_m - 1)m + O(e^4) \]

we get:

\[ \Sigma(\phi) = - \frac{e^2}{8\pi^2} \int_0^1 dx \left( (2-\varepsilon)(1-x)\phi + (4-\varepsilon)m \right) \left[ \frac{1}{\varepsilon^2} - \frac{1}{2} \ln(D/\mu^2) \right] \]

\[ - (Z_2 - 1)\phi - (Z_m - 1)m + O(e^4) \]

we set \( Z \)’s to cancel divergent parts

\[ Z_2 = 1 - \frac{e^2}{8\pi^2} \left( \frac{1}{\varepsilon^2} + \text{finite} \right) + O(e^4) \]

\[ Z_m = 1 - \frac{e^2}{2\pi^2} \left( \frac{1}{\varepsilon^2} + \text{finite} \right) + O(e^4) \]

we can impose \( \Sigma(-m) = 0 \) by writing:

\[ \Sigma(\phi) = \frac{e^2}{8\pi^2} \int_0^1 dx \left( (1-x)\phi + 2m \right) \ln(D/D_0) + \kappa_2(\phi + m) \]

\[ + O(e^4) \]

fixed by imposing: \( \Sigma'(-m) = 0 \)

\[ \kappa_2 = -2 \int_0^1 dx x(1-x^2)m^2/D_0 \]

\[ = -2 \ln(m/m_\gamma) + 1 \]
Let's start with the quark propagator:

\[
\frac{1}{p^2 - m^2} \cdot \frac{1}{(p + l)^2 - m^2} \cdot \frac{1}{i p^i} = - (Z_2 - 1) \delta_{ij} \psi
\]

the calculation is identical to QED with additional color factor:

\[
(T^a T^a)_{ij} = C(R) \delta_{ij}
\]

the result has to be identical to QED up to the color factor:

\[
Z_2 = 1 - C(R) \frac{g^2}{8\pi^2} \frac{1}{\varepsilon} + O(g^4)
\]

Finally, let's evaluate the diagram contributing to the vertex:

\[
iV_{ij}^{\mu} = ig^\mu \gamma^a T^a_{ij}
\]
combining denominators...

\[ iV_{1\text{loop}}^\mu(p',p) = (ie)^3 \left( \frac{1}{e} \right)^3 \int \frac{d^4q}{(2\pi)^4} \left[ \gamma^\rho \tilde{S}(\gamma^\nu + \gamma^\mu \gamma^\rho \gamma^\sigma ) \Delta_{\nu\rho}(e) \right] \]

\[ q = \ell + x_1p + x_2p', \]
\[ D = x_1(1-x_1)p^2 + x_2(1-x_2)p'^2 - 2x_1x_2p\cdot p' + (x_1 + x_2)m^2 + x_3m_\gamma^2, \]
\[ N^\mu = \gamma^\nu(-\gamma' - \ell + m)\gamma^\mu(-\gamma' - \ell + m)\gamma^\nu \]
\[ = \gamma^\nu[-\gamma' + x_1\gamma' + (1-x_1)\gamma' + m]\gamma^\mu[-\gamma' + (1-x_1)\gamma' + x_2\gamma' + m]\gamma^\nu \]
\[ \tilde{N}^\mu = \gamma^\nu[x_1\gamma' - (1-x_2)\gamma' + m]\gamma^\mu[-(1-x_1)\gamma' + x_2\gamma' + m]\gamma^\nu \]

\[ iV_{1\text{loop}}^\mu(p',p) = e^3 \int dF_3 \int \frac{d^4q}{(2\pi)^4} \frac{N^\mu}{(q^2 + D)^3} \]

\[ N^\mu = \gamma^\nu\gamma^\mu\gamma^\nu + \tilde{N}^\mu + \text{(linear in } q) \]

continuing to d dimensions

\[ \gamma^\nu\gamma^\mu\gamma^\nu \to \frac{1}{d} q^2 \gamma^\nu \gamma^\rho \gamma^\mu \gamma^\rho \gamma^\nu = \frac{(d-2)^2}{d} q^2 \gamma^\mu \]

evaluating the loop integral we get:

\[ V_{1\text{loop}}^\mu(p',p) = e^3 \frac{3}{8\pi^2} \left[ \left( \frac{1}{e} - 1 - \frac{1}{2} \int dF_3 \ln(D/\mu^2) \right) \gamma^\mu + \frac{1}{4} \int dF_3 \frac{\tilde{N}^\mu}{D} \right] \]

\[ iV^\mu(p',p) = iZ_1 e^\gamma^\mu + iV_{1\text{loop}}^\mu(p',p) + O(e^5) \]
the infinite part can be absorbed by Z

\[ Z_1 = 1 - \frac{e^2}{8\pi^2} \left( \frac{1}{e} + \text{finite} \right) + O(e^4) \]

the finite part of the vertex function is fixed by a suitable condition.
Let's continue with the quark-quark-gluon vertex:

\[
\begin{align*}
T^b T^a T^b &= T^b \left( T^b T^a + \frac{1}{2} i f^{abc} T^c \right) \\
&= C(R) T^a + \frac{1}{2} i f^{abc} [T^b, T^c] \\
&= C(R) T^a + \frac{1}{2} (i f^{abc}(i f^{bcd}) T^d) \\
&= C(R) T^a - \frac{1}{2} (T^a_{bc} T^d_{cd} T^d) \\
&= [C(R) - \frac{1}{2} T(A)] T^a .
\end{align*}
\]

thus the divergent part is the same as in QED up to the color factor: 

\[
\frac{g^2}{8\pi^2\epsilon} i g T^a_{ij} \gamma^\mu .
\]

Let's evaluate the second diagram:

\[
i V^{abc}_{\mu\nu\rho}(p, q, r) = g f^{abc} [(q-r)_{\mu} g_{\nu\rho} + (r-p)_{\nu} g_{\rho\mu} + (p-q)_{\rho} g_{\mu\nu}]
\]

the divergent piece doesn't depend on external momenta, so we can set them to 0; using Feynman rules we get:

\[
i V^{a\mu}_{ij}(0, 0) = (ig)^2 g f^{abc} (T^c T^b)_{ij} \left( \frac{1}{i} \right)^3 \int \frac{d^4 \ell}{(2\pi)^4} \frac{\gamma_{\mu}(-\ell+m)\gamma_{\nu}}{\ell^2 \ell^2 (\ell^2+m^2)} \\
\times [(\ell-(-\ell))^\mu g^{\nu\rho} + (-\ell-0)^\nu g^{\rho\mu} + (0-\ell)^\rho g^{\mu\nu}]
\]
\[ iV^{a\mu}_{ij}(0,0) = (ig)^2 g f^{abc}(T^c T^b)_{ij} \left(\frac{1}{i}\right)^3 \int \frac{d^4 \ell}{(2\pi)^4} \frac{\gamma_\rho(-\ell+m)\gamma_\nu}{\ell^2\ell^2(\ell^2+m^2)} \times \left[ (\ell-(-\ell))^{\mu} g^{\nu\rho} + (-\ell-0)^\nu g^{a\mu} + (0-\ell)^\rho g^{a\nu} \right] \]

\[ f^{abc} T^c T^b = \frac{1}{2} f^{abc}[T^c, T^b] \leq \frac{1}{2} i f^{abc} f^{cde} T^d \leq -\frac{1}{2} i T(A) T^a \]

the numerator is:

\[ N^\mu = \gamma_\rho (\gamma_\sigma \ell^\sigma + m) \gamma_\nu \left(2\ell^\mu g^{\nu\rho} - \ell^\nu g^{a\mu} - \ell^\rho g^{a\nu}\right) \]

\[ \ell^\sigma \ell^\mu \rightarrow d^{-1} \ell^2 g^{\sigma\mu} \]

we can drop linear terms

\[ N^\mu \rightarrow -d^{-1} \ell^2 \left(2\gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\mu \gamma_\nu (2g^{\sigma\mu} g^{\nu\rho} - g^{\sigma\nu} g^{a\mu} - g^{\sigma\rho} g^{a\nu}) \right) \]

\[ \left(2\gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\mu \gamma_\nu (2g^{\sigma\mu} g^{\nu\rho} - g^{\sigma\nu} g^{a\mu} - g^{\sigma\rho} g^{a\nu}) \right) \]

\[ \left(2\gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\mu \gamma_\nu (2g^{\sigma\mu} g^{\nu\rho} - g^{\sigma\nu} g^{a\mu} - g^{\sigma\rho} g^{a\nu}) \right) \]

\[ \rightarrow d^{-1} \ell^2 (2\gamma^\nu \gamma^\rho \gamma_\nu - \gamma^\mu \gamma^\nu \gamma_\rho \gamma_\nu - \gamma^\rho \gamma^\mu \gamma_\nu - \gamma_\nu \gamma^\sigma \gamma^\rho \gamma^\mu) \]

\[ \rightarrow d^{-1} \ell^2 (2(d-2) + \gamma^\mu \gamma^\nu \gamma_\mu \gamma_\nu) \]

for \( d = 4 \) (we are interested in the divergent part only):

\[ N^\mu \rightarrow -3\ell^2 \gamma^\mu \]

\[ iV^{a\mu}_{ij}(0,0) = (ig)^2 g f^{abc}(T^c T^b)_{ij} \left(\frac{1}{i}\right)^3 \int \frac{d^4 \ell}{(2\pi)^4} \frac{\gamma_\rho(-\ell+m)\gamma_\nu}{\ell^2\ell^2(\ell^2+m^2)} \times \left[ (\ell-(-\ell))^{\mu} g^{\nu\rho} + (-\ell-0)^\nu g^{a\mu} + (0-\ell)^\rho g^{a\nu} \right] \]

\[ f^{abc} T^c T^b = \frac{1}{2} f^{abc}[T^c, T^b] \leq \frac{1}{2} i f^{abc} f^{cde} T^d \leq -\frac{1}{2} i T(A) T^a \]

thus for the divergent part of the 2nd diagram we find:

\[ iV^{a\mu}_{ij}(0,0) = \frac{3}{2} T(A) g^3 T^a_{ij} \gamma^\mu \int \frac{d^4 \ell}{(2\pi)^4} \frac{1}{\ell^2(\ell^2+m^2)} \]

\[ \rightarrow \frac{i}{8\pi^2 \varepsilon} + O(\varepsilon^0) \]

putting pieces together we get:

\[ V^{a\mu}_{ij}(0,0)_{\text{div}} = \left(Z_1 + \left[C(R)-\frac{1}{2} T(A)\right] \frac{g^2}{8\pi^2 \varepsilon} + \frac{3}{2} T(A) \frac{g^2}{8\pi^2 \varepsilon}\right) g T^a_{ij} \gamma^\mu \]

and we find:

\[ Z_1 = 1 - \left[C(R) + T(A)\right] \frac{g^2}{8\pi^2} \frac{1}{\varepsilon^2} + O(g^4) \]

in Feynman gauge and the \( \overline{\text{MS}} \) scheme.
Let's now calculate the $i\Pi^{\mu\nu}(k)$ at one loop:

$$i\Pi^{\mu\nu}(k) = (-1)(iZ_1e)^2\left(\frac{1}{i}\right)^2 \int \frac{d^4l}{(2\pi)^4} \text{Tr} \left[ \tilde{S}(l+k)\gamma^\mu \tilde{S}(l)\gamma^\nu \right]$$

extra -1 for fermion loop; and the trace

$$\tilde{S}(\phi) = -\frac{\gamma + m}{p^2 + m^2 - i\epsilon}$$

$$- i(Z_3-1)(k^2g^{\mu\nu} - k^\mu k^\nu) + \mathcal{O}(e^4),$$

$$\quad - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} = -\frac{1}{2} \partial^\mu A^\nu \partial_\mu A_\nu + \frac{1}{2} \partial^\mu A^\nu \partial_\nu A_\mu$$

$$\mathcal{L}_{ct} = i(Z_2-1)\bar{\Psi}\gamma^\mu \Psi - (Z_3-1)m\bar{\Psi}\Psi - \frac{1}{4}(Z_3-1)F^{\mu\nu} F_{\mu\nu}$$

$$i\Pi^{\mu\nu}(k) = (-1)(iZ_1e)^2\left(\frac{1}{i}\right)^2 \int \frac{d^4l}{(2\pi)^4} \text{Tr} \left[ \tilde{S}(l+k)\gamma^\mu \tilde{S}(l)\gamma^\nu \right]$$

$$\quad - i(Z_3-1)(k^2g^{\mu\nu} - k^\mu k^\nu) + \mathcal{O}(e^4),$$

$$\tilde{S}(\phi) = -\frac{\gamma + m}{p^2 + m^2 - i\epsilon}$$

$$\text{Tr} \left[ \tilde{S}(l+k)\gamma^\mu \tilde{S}(l)\gamma^\nu \right] = \int_0^1 dx \frac{4N^{\mu\nu}}{(q^2 + D)^2}$$

$$q = \ell + xk$$

$$D = x(1-x)k^2 + m^2 - i\epsilon$$

$$4N^{\mu\nu} = \text{Tr} \left[ (-\ell-k+m)\gamma^\mu (-\ell+m)\gamma^\nu \right]$$

$$\text{Tr}[\delta^{abc}] = 4[ad(b)c - (ac)(bd) + (ab)(cd)]$$

$$\text{Tr}[\delta^{ab}] = -4(ab)$$

$$N^{\mu\nu} = (\ell+k)^\mu \ell^\nu + \ell^\mu (\ell+k)^\nu - [\ell(\ell+k) + m^2]g^{\mu\nu}$$

$$\ell = q - xk$$

$$N^{\mu\nu} \rightarrow 2q^\mu q^\nu - 2x(1-x)k^\mu k^\nu - [q^2 - x(1-x)k^2 + m^2]g^{\mu\nu}$$

we ignore terms linear in $q$
\[ i \Pi^{\mu\nu}(k) = (-1)(iZ_1e)^2 \left( \frac{1}{\pi} \right)^2 \int \frac{d^d q}{(2\pi)^d} \text{Tr} \left[ \hat{S}(f+\kappa) \gamma^\mu \hat{S}(f) \gamma^\nu \right] \]
\[ - i (Z_3-1)(k^2 g^{\mu\nu} - k^\mu k^\nu) + O(e^4) , \]
\[ \text{Tr} \left[ \hat{S}(f+\kappa) \gamma^\mu \hat{S}(f) \gamma^\nu \right] = \int_0^1 dx \frac{4N^{\mu\nu}}{(q^2 + D)^2} \]
\[ N^{\mu\nu} = 2q^\mu q^\nu - 2x(1-x)k^\mu k^\nu - [q^2 - x(1-x)k^2 + m^2]g^{\mu\nu} \]

the integral diverges in 4 spacetime dimensions and so we analytically continue it to \( d = 4-\varepsilon \); we also make the replacement \( e \to e \mu^{\varepsilon/2} \) to keep the coupling dimensionless:
\[ \int d^d q \ q^\mu q^\nu f(q^2) = \frac{1}{d} g^{\mu\nu} \int d^d q \ q^2 f(q^2) \]
\[ N^{\mu\nu} \to 2x(1-x)(k^2 g^{\mu\nu} - k^\mu k^\nu) \]

is transverse :)

\[ i \Pi^{\mu\nu}(k) = (-1)(iZ_1e)^2 \left( \frac{1}{\pi} \right)^2 \int \frac{d^d q}{(2\pi)^d} \text{Tr} \left[ \hat{S}(f+\kappa) \gamma^\mu \hat{S}(f) \gamma^\nu \right] \]
\[ - i (Z_3-1)(k^2 g^{\mu\nu} - k^\mu k^\nu) + O(e^4) , \]
\[ \text{Tr} \left[ \hat{S}(f+\kappa) \gamma^\mu \hat{S}(f) \gamma^\nu \right] = \int_0^1 dx \frac{4N^{\mu\nu}}{(q^2 + D)^2} \]

the integral over \( q \) is straightforward:
\[ \mu^2 \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + D)^2} = \frac{i}{16\pi^2} \Gamma \left( \frac{d}{2} \right) \Gamma \left( \frac{d}{2} + 1 \right) \frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d}{2} + 1)} = \frac{i}{8\pi^2} \left[ \frac{1}{\varepsilon - \frac{1}{2} \ln(D/\mu^2)} \right] \]
\[ \mu^2 = 4\pi e^{-\gamma} \mu^2 \]
\[ \Pi(k^2) = - \frac{e^2}{\pi^2} \int_0^1 dx x(1-x) \left[ \frac{1}{\varepsilon} - \frac{1}{2} \ln(D/\mu^2) \right] - (Z_3-1) + O(e^4) \]

imposing \( \Pi(0) = 0 \) fixes
\[ Z_3 = 1 - \frac{e^2}{6\pi^2} \left[ \frac{1}{\varepsilon} - \ln(m/\mu) \right] + O(e^4) \]

and
\[ \Pi(k^2) = \frac{e^2}{2\pi^2} \int_0^1 dx x(1-x) \ln(D/m^2) + O(e^4) \]
Let's now evaluate one-loop corrections to the gluon propagator:

\[ i
\]
\[ i \Pi^{\mu \nu ab}(k) = \frac{1}{2} g^2 f^{acd} f^{bcd} \left( \frac{1}{i} \right)^2 \int \frac{d^4 \ell}{(2\pi)^4} \frac{\mathcal{N}^{\mu \nu}}{\ell^2 (\ell + k)^2} \]

\[ f^{acd} f^{bcd} = T(A) \delta^{ab} \]

\[ \mathcal{N}^{\mu \nu} = -[(2\ell + k) \rho^\rho - (\ell - k)^\rho g^{\sigma \mu} - (\ell + 2k)^\sigma g^{\mu \rho}] \times [(2\ell + k)^\nu g_{\rho \sigma} - (\ell - k)_\rho \delta_\sigma^\nu - (\ell + 2k)_\sigma \delta^\nu_{\rho}] \]

Combining denominators and continuing to \( d = 4 - \varepsilon \) dimension:

\[ -\frac{1}{2} g^2 T(A) \delta^{ab} \mu^\varepsilon \int_0^1 dx \int \frac{d^d q}{(2\pi)^d} \frac{\mathcal{N}^{\mu \nu}}{(q^2 + D)^2} \]

\[ q = \ell + xk \]

\[ D = x(1-x)k^2 \]

where

\[ \mathcal{N}^{\mu \nu} = -[(2q + (1-2x)k) \mu g^{\rho \sigma} - (q - (1+x)k) \rho g^{\sigma \mu} - (q + (2-x)k) \sigma g^{\mu \rho}] \times [(2q + (1-2x)k) \nu g_{\rho \sigma} - (q - (1+x)k) \rho \delta_\sigma^\nu - (q + (2-x)k) \sigma \delta^\nu_{\rho}] \]

Expanding and ignoring terms linear in \( q \):

\[ \mathcal{N}^{\mu \nu} \rightarrow -2q^2 g^{\mu \nu} - (4d - 6)q^{\mu} q^{\nu} \]

\[ - [(1+x)^2 + (2-x)^2] k^2 g^{\mu \nu} \]

\[ - [d(1-2x)^2 + 2(1-2x)(1+x) - 2(2-x)(1+x) - 2(2-x)(1-2x)] k^\mu k^\nu \]

We are interested in the divergent part only:

\[ \mathcal{N}^{\mu \nu} \rightarrow -\frac{9}{2} q^2 g^{\mu \nu} - (5-2x+2x^2) k^2 g^{\mu \nu} + (2+10x-10x^2) k^\mu k^\nu \]
integrating over \( x \), we get the result for the divergent part of

\[
- \frac{ig^2}{16\pi^2} T(A) \delta^{ab} \left( -\frac{19}{6} k^2 g^{\mu\nu} + \frac{11}{3} k^\mu k^\nu \right)
\]
finally, let’s calculate the fermion loop:

\[ - \frac{ig^2}{6\pi^2} n_F T(R) \delta^{ab} \frac{1}{\varepsilon} \left( k^2 g^{\mu\nu} - k^\mu k^\nu \right) \]

the same calculation as in QED, except for the color factor:

\[ \text{Tr}(T^a T^b) = T(R) \delta^{ab} \]

the number of flavors

Putting pieces together:

we find:

\[ \Pi^{\mu\nu ab}(k) = \Pi(k^2)(k^2 g^{\mu\nu} - k^\mu k^\nu) \delta^{ab} \]

gluon self-energy is transverse

\[ \Pi(k^2)_{\text{div}} = -(Z_3 - 1) + \left[ \frac{5}{3} T(A) - \frac{4}{3} n_F T(R) \right] \frac{g^2}{8\pi^2} \frac{1}{\varepsilon} + O(g^4) \]

and so:

\[ Z_3 = 1 + \left[ \frac{5}{3} T(A) - \frac{4}{3} n_F T(R) \right] \frac{g^2}{8\pi^2} \frac{1}{\varepsilon} + O(g^4) \]
We found: in Feynman gauge and the $\overline{\text{MS}}$ scheme

\[
\begin{align*}
Z_1 &= 1 - \left[ C(R) + T(A) \right] \frac{g^2}{8\pi^2} \frac{1}{\varepsilon} + O(g^4), \\
Z_2 &= 1 - C(R) \frac{g^2}{8\pi^2} \frac{1}{\varepsilon} + O(g^4), \\
Z_3 &= 1 + \left[ \frac{5}{3} T(A) - \frac{4}{3} n_f T(R) \right] \frac{g^2}{8\pi^2} \frac{1}{\varepsilon} + O(g^4),
\end{align*}
\]

not equal!

Let's calculate the beta function; define:

\[
\alpha \equiv \frac{g^2}{4\pi}
\]

the dictionary:

\[
\begin{align*}
\alpha_0 &= \frac{Z_1^2}{Z_2^2 Z_3} \alpha \bar{\mu}^\varepsilon \\
g_0^2 &= \frac{Z_1^2 Z_2^2 g^2 \bar{\mu}^\varepsilon}{Z_2^2 Z_3} \\
g_0^2 &= \frac{Z_3}{Z_3} \bar{g}^2 \bar{\mu}^\varepsilon \\
g_0^2 &= \frac{Z_3}{Z_3} \bar{g}^2 \bar{\mu}^\varepsilon
\end{align*}
\]

**Beta functions in quantum electrodynamics based on S-66**

Let's calculate the beta function in QED:

\[
\begin{align*}
\mathcal{L}_0 &= i \bar{\Psi} i \not{\partial} \Psi - m \bar{\Psi} \Psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \\
\mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_1 \\
\mathcal{L}_1 &= Z_1 e \bar{\Psi} \not{A} \Psi + \mathcal{L}_{\text{ct}} , \\
\mathcal{L}_{\text{ct}} &= i (Z_2 - 1) \bar{\Psi} i \not{\partial} \Psi - (Z_m - 1) m \bar{\Psi} \Psi - \frac{1}{4} (Z_3 - 1) F^{\mu\nu} F_{\mu\nu}
\end{align*}
\]

the dictionary:

\[
\begin{align*}
e_0 &= \frac{1}{2} Z_2^{-1/2} Z_1^{-1/2} \bar{\mu}^\varepsilon/2 \varepsilon \\
\alpha &= e^2/4\pi \\
\alpha_0 &= \frac{Z_3}{Z_2^2 Z_1} \bar{\mu}^\varepsilon \alpha
\end{align*}
\]

\[
\begin{align*}
Z_1 &= 1 - \frac{\alpha}{2\pi} \frac{1}{\varepsilon} + O(\alpha^2) \\
Z_2 &= 1 - \frac{\alpha}{2\pi} \frac{1}{\varepsilon} + O(\alpha^2) \\
Z_3 &= 1 - \frac{2\alpha}{3\pi} \frac{1}{\varepsilon} + O(\alpha^2)
\end{align*}
\]

Note $Z_1 = Z_2$! 
following the usual procedure:

\[ \alpha_0 = Z_3^{-1} Z_2^{-2} Z_1^2 \mu \varepsilon \alpha \]

\[ \ln \alpha_0 = \sum_{n=1}^{\infty} \frac{E_n(\alpha)}{\varepsilon^n} + \ln \alpha + \varepsilon \ln \tilde{\mu} \]

\[ \ln \left( Z_3^{-1} Z_2^{-2} Z_1^2 \right) = \sum_{n=1}^{\infty} \frac{E_n(\alpha)}{\varepsilon^n} \]

\[ Z_1 = 1 - \frac{\alpha}{2\pi \varepsilon} + O(\alpha^2) \]
\[ Z_2 = 1 - \frac{\alpha}{2\pi \varepsilon} + O(\alpha^2) \]
\[ Z_3 = 1 - \frac{2\alpha}{3\pi \varepsilon} + O(\alpha^2) \]

we find:

\[ \beta(\alpha) = \frac{2\alpha^2}{3\pi} + O(\alpha^3) \]

or equivalently:

\[ \beta(e) = \frac{e^3}{12\pi^2} + O(e^5) \]

\[ \alpha = \frac{e^2}{4\pi} \]
\[ \dot{\alpha} = \frac{e\dot{e}}{2\pi} \]

For a theory with \( N \) Dirac fields with charges \( Q_i e \):

\[ Z_1 = 1 - \frac{\alpha}{2\pi \varepsilon} + O(\alpha^2) \]
\[ Z_2 = 1 - \frac{\alpha}{2\pi \varepsilon} + O(\alpha^2) \]
\[ Z_3 = 1 - \frac{2\alpha}{3\pi \varepsilon} + O(\alpha^2) \]

we find:

\[ \beta(e) = \frac{\sum_{i=1}^{N} Q_i^2}{12\pi^2} e^3 + O(e^5) \]
following the usual procedure:

\[
\alpha_0 = Z_3^{-1}Z_2^{-2}Z_1^2 \, \mu^\varepsilon \alpha
\]

\[
\ln \alpha_0 = \sum_{n=1}^{\infty} \frac{G_n(\alpha)}{\varepsilon^n} + \ln \alpha + \varepsilon \ln \tilde{\mu}
\]

\[
\ln \left( Z_3^{-1}Z_2^{-2}Z_1^2 \right) = \sum_{n=1}^{\infty} \frac{G_n(\alpha)}{\varepsilon^n}
\]

\[
Z_1 = 1 - \left[ C(R) + T(A) \right] \frac{g^2}{8\pi^2} \frac{1}{\varepsilon} + O(g^4),
\]

\[
Z_2 = 1 - C(R) \frac{g^2}{8\pi^2} \frac{1}{\varepsilon} + O(g^4),
\]

\[
Z_3 = 1 + \left[ \frac{2}{3} T(A) - \frac{4}{3} n_F T(R) \right] \frac{g^2}{8\pi^2} \frac{1}{\varepsilon} + O(g^4),
\]

we find:

\[
\beta(\alpha) = \alpha^2 G_1'(\alpha)
\]

\[
\beta(\alpha) = -\left[ \frac{11}{3} T(A) - \frac{4}{3} n_F T(R) \right] \frac{\alpha^2}{2\pi} + O(\alpha^3)
\]

\[
\beta(\alpha) = -\left[ \frac{11}{3} T(A) - \frac{4}{3} n_F T(R) \right] \frac{\alpha^2}{2\pi} + O(\alpha^3)
\]

\[
\alpha = \frac{g^2}{4\pi}
\]

\[
\dot{\alpha} = \frac{g g}{2\pi}
\]

or equivalently:

\[
\beta(g) = -\left[ \frac{11}{3} T(A) - \frac{4}{3} n_F T(R) \right] \frac{g^3}{16\pi^2} + O(g^5)
\]

For QCD:

\[
T(A) = 3
\]

\[
T(R) = \frac{1}{2}
\]

\[
11 - \frac{2}{3} n_F
\]

beta function is negative for \( n_F \leq 16 \),
the gauge coupling gets weaker at higher energies!
BRST symmetry

We are going to show that the gauge-fixed lagrangian:

$$\mathcal{L} \equiv \mathcal{L}_{YM} + \mathcal{L}_{gf} + \mathcal{L}_{gh}$$

has a residual form of the gauge symmetry - Becchi-Rouet-Stora-Tyutin symmetry

Consider an infinitesimal transformation for a non-abelian gauge theory:

$$\delta A^a_{\mu}(x) = -D^{ab}_{\mu}b^b(x)$$

$$\delta \phi_i(x) = -ig\theta^a(x)(T^a_{R})_{ij}\phi_j(x)$$

The BRST transformation is defined as:

$$\delta_B A^a_{\mu}(x) \equiv D^{ab}_{\mu}c^b(x)$$

$$= \partial_{\mu}c^a(x) - gf^{abc}A^c_{\mu}(x)c^b(x)$$

we use the ghost field (scalar Grassmann field) instead of $-\theta^a(x)$.

$$\delta_B \phi_i(x) \equiv igc^a(x)(T^a_{R})_{ij}\phi_j(x)$$

Anything that is gauge invariant is automatically BRST invariant, in particular $\delta_B \mathcal{L}_{YM} = 0$ !

Now we are going to require that $\delta_B \delta_B = 0$:

this requirement will determine the BRST transformation of the ghost field.

$$\delta_B \phi_i(x) = ig(\delta_B c^a(x)(T^a_{R})_{ij}\phi_j(x)$$

$$\delta_B(\delta_B \phi_i) = ig(\delta_B c^a(x)(T^a_{R})_{ij}\phi_j(x) - igc^a(T^a_{R})_{ij}\delta_B \phi_j$$

-1 for $\delta_B$ acting as an anticommuting object

$$\delta_B(\delta_B \phi_i) = ig(\delta_B c^a(x)(T^a_{R})_{ij}\phi_j(x) - g^2 c^a c^b(T^a_{R}T^b_{R})_{ik}\phi_k$$

$$c^b c^a = -c^a c^b$$

$$\frac{1}{2}[T^a_{R}, T^b_{R}] = \frac{i}{2}f^{abc}T^c_{R}$$

Thus we have:

$$\delta_B(\delta_B \phi_i) = ig(\delta_B c^c + \frac{1}{2}gf^{abc}c^a c^b)(T^c_{R})_{ij}\phi_j$$

that will vanish for all $\phi_j(x)$ if and only if:

$$\delta_B c^c(x) = -\frac{1}{2}gf^{abc}c^a(x)c^b(x)$$
Now we have to check that $\delta_B \delta_B = 0$ for the gauge field:

$$\delta_B A^a_{\mu}(x) \equiv D^a_{\mu} c^b(x) = \partial_\mu c^a(x) - g f^{abc} A^c_{\mu}(x) c^b(x)$$

$$\delta_B (\delta_B A^a_{\mu}) = (\delta_B \partial_\mu - g f^{abc} A^c_{\mu} c^b)(\delta_B c^b) - g f^{abc} (\delta_B A^c_{\mu}) c^b$$

$$= D^a_{\mu} (\delta_B c^b) - g f^{abc} (D^d_{\mu} c^d) c^b$$

$$= D^a_{\mu} (\delta_B c^b) - g f^{abc} (\partial_\mu c^c) c^b + g^2 f^{abc} f^{dec} A^e_{\mu} c^d c^b$$

$$\frac{1}{2} (\partial_\mu c^c) c^b \equiv \frac{1}{2} (\partial_\mu c^c) c^b - \frac{1}{2} (\partial_\mu c^b) c^c$$

$$= \frac{1}{2} (\partial_\mu c^c) c^b + \frac{1}{2} c^c (\partial_\mu c^b)$$

$$= \frac{1}{2} \partial_\mu (c^c c^b) .$$

Now we have to check that $\delta_B \delta_B = 0$ for the gauge field:

$$\delta_B A^a_{\mu}(x) \equiv D^a_{\mu} c^b(x) = \partial_\mu c^a(x) - g f^{abc} A^c_{\mu}(x) c^b(x)$$

$$\delta_B (\delta_B A^a_{\mu}) = (\delta_B \partial_\mu - g f^{abc} A^c_{\mu} c^b)(\delta_B c^b) - g f^{abc} (\delta_B A^c_{\mu}) c^b$$

$$= D^a_{\mu} (\delta_B c^b) - g f^{abc} (D^d_{\mu} c^d) c^b$$

$$= D^a_{\mu} (\delta_B c^b) - g f^{abc} (\partial_\mu c^c) c^b + g^2 f^{abc} f^{dec} A^e_{\mu} c^d c^b$$

$$\frac{1}{2} (\partial_\mu c^c) c^b \equiv \frac{1}{2} (\partial_\mu c^c) c^b - \frac{1}{2} (\partial_\mu c^b) c^c$$

$$= \frac{1}{2} (\partial_\mu c^c) c^b + \frac{1}{2} c^c (\partial_\mu c^b)$$

$$= \frac{1}{2} \partial_\mu (c^c c^b) .$$

Now we have to check that $\delta_B \delta_B = 0$ for the gauge field:

$$\delta_B A^a_{\mu}(x) \equiv D^a_{\mu} c^b(x) = \partial_\mu c^a(x) - g f^{abc} A^c_{\mu}(x) c^b(x)$$

$$\delta_B (\delta_B A^a_{\mu}) = (\delta_B \partial_\mu - g f^{abc} A^c_{\mu} c^b)(\delta_B c^b) - g f^{abc} (\delta_B A^c_{\mu}) c^b$$

$$= D^a_{\mu} (\delta_B c^b) - g f^{abc} (D^d_{\mu} c^d) c^b$$

$$= D^a_{\mu} (\delta_B c^b) - g f^{abc} (\partial_\mu c^c) c^b + g^2 f^{abc} f^{dec} A^e_{\mu} c^d c^b$$

$$\frac{1}{2} (\partial_\mu c^c) c^b \equiv \frac{1}{2} (\partial_\mu c^c) c^b - \frac{1}{2} (\partial_\mu c^b) c^c$$

$$= \frac{1}{2} (\partial_\mu c^c) c^b + \frac{1}{2} c^c (\partial_\mu c^b)$$

$$= \frac{1}{2} \partial_\mu (c^c c^b) .$$

Now we have to check that $\delta_B \delta_B = 0$ for the gauge field:

$$\delta_B A^a_{\mu}(x) \equiv D^a_{\mu} c^b(x) = \partial_\mu c^a(x) - g f^{abc} A^c_{\mu}(x) c^b(x)$$

$$\delta_B (\delta_B A^a_{\mu}) = (\delta_B \partial_\mu - g f^{abc} A^c_{\mu} c^b)(\delta_B c^b) - g f^{abc} (\delta_B A^c_{\mu}) c^b$$

$$= D^a_{\mu} (\delta_B c^b) - g f^{abc} (D^d_{\mu} c^d) c^b$$

$$= D^a_{\mu} (\delta_B c^b) - g f^{abc} (\partial_\mu c^c) c^b + g^2 f^{abc} f^{dec} A^e_{\mu} c^d c^b$$

$$\frac{1}{2} (\partial_\mu c^c) c^b \equiv \frac{1}{2} (\partial_\mu c^c) c^b - \frac{1}{2} (\partial_\mu c^b) c^c$$

$$= \frac{1}{2} (\partial_\mu c^c) c^b + \frac{1}{2} c^c (\partial_\mu c^b)$$

$$= \frac{1}{2} \partial_\mu (c^c c^b) .$$

vanishes for the variation of the ghost field we found before:

$$\delta_B c^c(x) = -\frac{1}{2} g f^{abc} c^a(x) c^b(x)$$
The BRST transformation of the antighost field is defined as:

\[ \delta_B \bar{c}^a(x) = B^a(x) \]

we treat ghost and antighost fields as independent fields

\[ \delta_B B^a(x) = 0 \]

B is a scalar field

Lautrup-Nakanishi auxiliary field

then \( \delta_B \delta_B = 0 \) implies:

What is it good for?

We can add to the lagrangian any term that is the BRST variation of some object:

\[ \mathcal{L} = \mathcal{L}_{YM} + \delta_B \mathcal{O} \]

\( \mathcal{L}_{YM} \) is BRST invariant because it is gauge invariant

we treat ghost and antighost fields as independent fields

\( \delta_B (\delta_B \mathcal{O}) = 0 \)

BRST invariant because corresponds to fixing a gauge

Let’s choose:

\[ \mathcal{O}(x) = \bar{c}^a(x) \left[ \frac{1}{2} \xi B^a(x) - G^a(x) \right] \]

\( \mathcal{O}(x) \) is BRST invariant because it is gauge invariant

\( \delta_B \mathcal{O} = (\delta_B \bar{c}^a) \left[ \frac{1}{2} \xi B^a (x) - \partial^\mu A_\mu^a (x) \right] - \bar{c}^a \left[ \frac{1}{2} \xi (\delta_B B^a (x)) - \partial^\mu (\delta_B A_\mu^a (x)) \right] \)

\( \delta_B A_\mu^a (x) = D_\mu^{ab} \bar{c}^b(x) \)

\( \delta_B \bar{c}^a(x) = B^a(x) \)

- 1 for \( \delta_B \) acting as an anticommuting object

\[ \delta_B \mathcal{O} = \frac{1}{2} \xi B^a B^a - B^a \partial^\mu A_\mu^a + \bar{c}^a \partial^\mu D_\mu^{ab} \bar{c}^b \]

or

\[ \delta_B \mathcal{O} \rightarrow \frac{1}{2} \xi B^a B^a - B^a \partial^\mu A_\mu^a - \partial^\mu \bar{c}^a D_\mu^{ab} \bar{c}^b \]
\[ \delta_B \mathcal{O} \rightarrow \frac{1}{2} \xi B^a B^a - B^a \partial^\mu A^a_{\mu} - \partial^\mu \bar{c}^a D^{ab}_{\mu} c^b \]

now we can easily perform the path integral over B:

it is equivalent to solving the classical equation of motion,

\[ \frac{\partial (\delta_B \mathcal{O})}{\partial B^a(x)} = \xi B^a(x) - \partial^\mu A^a_{\mu}(x) = 0 \]

and substituting the result back to the formula:

\[ \delta_B \mathcal{O} \rightarrow -\frac{1}{2} \xi^{-1} \partial^\mu A^a_{\mu} \partial^\nu A^a_{\nu} - \partial^\mu \bar{c}^a D^{ab}_{\mu} c^b \]

we obtained the gauge fixing lagrangian and the ghost lagrangian

\[
\mathcal{L} = \mathcal{L}_{YM} + \delta_B \mathcal{O}
\]

\[ S = \int d^4x \mathcal{L} \]

Symmetries of the complete action:

- Lorentz invariance
- Parity, Time reversal, Charge conjugation
- Global invariance under a given (non-abelian) symmetry group
- BRST invariance
- ghost number conservation (+1 for ghost and -1 for antighost)
- antighost translation invariance

\[ \bar{c}^a(x) \rightarrow \bar{c}^a(x) + \chi \]

The lagrangian already includes all the terms allowed by these symmetries!

this means that all the divergencies can be absorbed by the Zs of these terms, BRST symmetry requires that the gauge coupling renormalize in the same way in each term.
There is the Noether current associated with the BRST symmetry:

\[ j^\mu_B(x) = \sum_I \frac{\partial L}{\partial (\partial_\mu \Phi_I(x))} \delta_B \Phi_I(x) \]

and the corresponding BRST charge:

\[ Q_B = \int d^3x \, j^0_B(x) \]

The BRST charge generates a BRST transformation:

\[ i[Q_B, A^a_\mu(x)] = D^{ab}_\mu c^b(x), \quad \delta_B A^a_\mu(x) = D^{ab}_\mu c^b(x) \]

\[ i\{Q_B, c^a(x)\} = -\frac{1}{2} g f^{abc} c^b(x) c^c(x), \quad \delta_B c^a(x) = -\frac{1}{2} g f^{abc} c^b(x) c^c(x) \]

\[ i\{Q_B, \bar{c}^a(x)\} = B^a(x), \quad \delta_B \bar{c}^a(x) = B^a(x) \]

\[ i[Q_B, B^a(x)] = 0, \quad \delta_B B^a(x) = 0 \]

\[ i[Q_B, \phi_i(x)]_+ = ig c^a(x) (T^a_R)_{ij} \phi_j(x), \quad \delta_B \phi_i(x) = ig c^a(x) (T^a_R)_{ij} \phi_j(x) \]

The energy-momentum four-vector is:

\[ P^\mu = \int d^3x \, T^{0\mu}(x) \]

Recall, we defined the space-time translation operator

\[ T(a) \equiv \exp(-i P^\mu a_\mu) \]

so that

\[ T(a)^{-1} \varphi_a(x) T(a) = \varphi_a(x - a) \]

we can easily verify it; for an infinitesimal transformation it becomes:

\[ [\varphi_a(x), P^\mu] = \frac{1}{i} \partial^\mu \varphi_a(x) \]

it is straightforward to verify this by using the canonical commutation relations for \( \varphi_a(x) \) and \( \Pi_a(x) \).
Since $\delta_B \delta_B = 0$ we have:

\[ Q_B^2 = 0 \]

Consider states for which:

\[ Q_B |\psi\rangle = 0 \]

such states are said to be in the Kernel of $Q_B$.

\begin{itemize}
  \item **Cohomology of $Q_B$:**
    \[ |\psi\rangle \neq Q_B |\chi\rangle \]
    
    we identify two states as a single element of the cohomology if their difference is in the image:
    \[ |\psi'\rangle = |\psi\rangle + Q_B |\zeta\rangle \]

  \item **Image of $Q_B$:**
    \[ |\psi\rangle = Q_B |\chi\rangle \]
    
    zero norm states:
    \[ \langle \psi | \psi \rangle = \langle \psi | Q_B |\chi\rangle = 0 \]
    \[ Q \text{ is hermitian: } \langle \psi | Q_B = 0. \]
\end{itemize}

Consider a normalized state in the cohomology:

\[ \langle \psi | \psi \rangle = 1, \ Q_B |\psi\rangle = 0, \ |\psi\rangle \neq Q_B |\chi\rangle \]

since the lagrangian is BRST invariant:

\[ [H, Q_B] = 0 \]

and so the time evolved state is still annihilated by $Q_B$:

\[ Q_B e^{-iHt} |\psi\rangle = e^{-iHt} Q_B |\psi\rangle = 0 \]

(in addition, a unitary time evolution does not change the norm of a state)

the time-evolved state must still be in the cohomology!

We will see shortly that the physical states of the theory correspond to the cohomology of $Q_B$!
All states in the theory can be generated from creation operators (we start with widely separated wave packets, and so we can neglect interaction):

\[ A^\mu(x) = \sum_{\lambda>\gamma} \int \frac{dk}{k} \left[ \epsilon^\mu_{\lambda}(k) a_{\lambda}(k) e^{ikx} + \epsilon^\mu_{\gamma}(k) a^\dagger_{\gamma}(k) e^{-ikx} \right] \]

\[ c(x) = \int \frac{dk}{k} \left[ c(k) e^{ikx} + c^\dagger(k) e^{-ikx} \right], \]

\[ \bar{c}(x) = \int \frac{dk}{k} \left[ b(k) e^{ikx} + b^\dagger(k) e^{-ikx} \right], \]

\[ \phi(x) = \int \frac{dk}{k} \left[ a_\phi(k) e^{ikx} + a^\dagger_\phi(k) e^{-ikx} \right], \]

we are neglecting group indices

represents matter fields

for \( k^\mu = (\omega, k) = \omega (1, 0, 0, 1) \) four polarization vectors can be chosen as:

\[ \epsilon^\mu_\omega(k) = \frac{1}{\sqrt{2}} (1, 0, 0, 1), \]

\[ \epsilon^\mu_\omega(k) = \frac{1}{\sqrt{2}} (1, 0, 0, -1), \]

\[ \epsilon^\mu_+(k) = \frac{1}{\sqrt{2}} (0, 1, -i, 0), \]

\[ \epsilon^\mu_-(k) = \frac{1}{\sqrt{2}} (0, 1, +i, 0). \]

Setting \( g = 0 \) and matching coefficients of \( e^{-ikx} \) we find:

\[ i[Q_B, A^a_\mu(x)] = D^{ab}_\mu(\epsilon^b_c(x)), \]

\[ i\{Q_B, c^a(x)\} = -\frac{1}{2} g f^{abc} c^b(x) c^c(x), \]

\[ i\{Q_B, \bar{c}^a(x)\} = B^a(x), \]

\[ i[Q_B, B^a(x)] = 0, \]

\[ i[Q_B, \phi_i(x)] = ig e^a(x)(T^a_R)_{ij} \phi_j(x). \]

we also use EM to eliminate B:

\[ \frac{\partial(\delta_{\mu}\mathcal{O})}{\partial B^a(x)} = \xi B^a(x) - \partial^\mu A^a_\mu(x) = 0 \]

\[ [Q_B, \bar{a}^\dagger_\lambda(k)] = \sqrt{2} \omega \delta_{\lambda>\gamma} c^\dagger(k), \]

\[ \{Q_B, c^\dagger(k)\} = 0, \]

\[ \{Q_B, b^\dagger(k)\} = \xi^{-1} \sqrt{2} \omega a^\dagger_< (k), \]

\[ [Q_B, a^\dagger_\phi(k)] = 0. \]
Consider a normalized state in the cohomology: \( \langle \psi | \psi \rangle = 1, Q_B | \psi \rangle = 0 \),

the state \( c^\dagger(k) | \psi \rangle \) is proportional to \( Q_B a^\dagger_\lambda(k) | \psi \rangle \) and so it is not in the cohomology

\[
[Q_B, a^\dagger_\lambda(k)] = \sqrt{2} \omega \delta_{\lambda \lambda'} c^\dagger(k),
\]

\[
\{Q_B, c^\dagger(k)\} = 0,
\]

\[
\{Q_B, b^\dagger(k)\} = \xi^{-1} \sqrt{2} \omega a^\dagger_<(k),
\]

\[
[Q_B, a^\dagger_<(k)] = 0.
\]

if we add a photon with polarization \(+\), the state \( a^\dagger_>(k) | \psi \rangle \) is not annihilated by \( Q \) and so it is not in the cohomology

the state \( b^\dagger(k) | \psi \rangle \) is not annihilated by \( Q \) and so it is not in the cohomology

states: \( a^\dagger_+(k) | \psi \rangle \), \( a^\dagger_-(k) | \psi \rangle \) and \( a^\dagger_\phi(k) | \psi \rangle \) are annihilated by \( Q \) but cannot be written as \( Q \) acting on some state and so they are in the cohomology!

the vacuum is also in the cohomology

Thus we found:

we can build an initial state of widely separated particles that is in the cohomology only with matter particles and photons with polarizations \(+\) and \(-\). No ghosts or \(+, <\) polarized photons can be produced in the scattering process (a state in the cohomology will evolve to another state in the cohomology).