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ATOMIC SUPERSYMMETRY, OSCILLATORS, AND THE PENNING TRAP

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INTRODUCTION

It is a pleasure to participate in celebrating Franco Iachello's 50th birthday. I wish him many happy returns.

At least two physical supersymmetries are known to exist in nature. One, discovered by Franco and his coworkers, makes connections between different nuclei [1]. The other, atomic supersymmetry, is the subject of part of this paper. It can be viewed as a symmetry-based approach to the construction of an effective central-potential model describing the behavior of the valence electron in atoms and ions.

This paper begins with some background information and a summary of results in atomic supersymmetry. The connection between the supersymmetric Coulomb and oscillator problems in arbitrary dimensions is outlined. Next, I treat the issue of finding a description of supersymmetry-based quantum-defect theory in terms of oscillators. A model with an anharmonic term that yields analytical eigenfunctions is introduced to solve this problem in arbitrary dimensions. Finally, I show that geonium atoms (particles contained in a Penning trap) offer a realization of a multidimensional harmonic oscillator in an idealized limit. The anharmonic theory presented here provides a means of modeling the realistic case.

SUPERSYMMETRIC QUANTUM MECHANICS

This section provides some background in supersymmetric quantum mechanics [2] and establishes notation.

A quantum-mechanical hamiltonian H_S is said to be supersymmetric if it commutes with N supersymmetry operators Q_j and if it is generated by anticommutators according to

$$\{Q_j, Q_k\} = \delta_{jk} H_S \quad . \tag{1}$$

The operators H_S and Q_j form the generators of a superalgebra denoted by sqm(N).

For the purposes of this paper it suffices to consider the special case $N = 2$, with superalgebra sqm(2). Define the linear combinations

$$Q = \sqrt{\tfrac{1}{2}}\,(Q_1 + iQ_2) \quad , \quad Q^\dagger = \sqrt{\tfrac{1}{2}}\,(Q_1 - iQ_2) \quad . \tag{2}$$

Then, the supersymmetric hamiltonian can be written

$$H_S = \{Q, Q^\dagger\} \quad . \tag{3}$$

For one-dimensional quantum systems, the superalgebra sqm(2) admits a two-dimensional representation. Write

$$Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \quad , \quad H_S = \begin{pmatrix} h_+ & 0 \\ 0 & h_- \end{pmatrix} \quad , \tag{4}$$

with

$$A = -i\partial_x - iU'/2 \quad . \tag{5}$$

Here, U' denotes dU/dx for some function $U = U(x)$. In this representation, the supersymmetric hamiltonian H_S contains two components h_+ and h_-, referred to as the bosonic and fermionic hamiltonians, respectively. These satisfy the equations

$$h_\pm \Psi_{\pm n} \equiv \big[-\frac{d^2}{dx^2} + V_\pm(x)\big]\Psi_{\pm n} = \epsilon_n \Psi_{\pm n}\,, \tag{6}$$

with

$$V_\pm(x) = (\tfrac{1}{2}U')^2 \mp \tfrac{1}{2}U'' \quad . \tag{7}$$

Using the above relations, some general properties of the supersymmetric system can be obtained. First, for unbroken supersymmetry the ground-state energy is zero. Second, except for the ground state (which appears in the spectrum of h_+) the bosonic and fermionic spectra are degenerate. Finally, the supersymmetry generators $Q, Q^\dagger$ map degenerate states from the two sectors into each other.

ATOMIC SUPERSYMMETRY: EXACT LIMIT

Consider the Schrödinger equation for the hydrogen atom. In spherical polar coordinates, the equation separates into angular and radial parts. The angular part gives the spherical harmonics, while the radial part can be expressed as

$$\big[-\frac{d^2}{dy^2} - \frac{1}{y} + \frac{l(l+1)}{y^2} - \frac{1}{2}E_n\big]\chi_{nl}(y) = 0 \quad . \tag{8}$$

Here, atomic units are used, and

$$y = 2r \quad , \quad E_n = -\frac{1}{2n^2} \quad , \quad \chi_{nl}(2r) = rR_{nl}(r) \quad . \tag{9}$$

The radial wave functions $R_{nl}(r)$ are given by

$$R_{nl}(r) = \frac{2}{n^2}\Big[\frac{\Gamma(n-l)}{\Gamma(n+l+1)}\Big]^{\frac{1}{2}}\Big(\frac{2r}{n}\Big)^l \exp\Big(-\frac{r}{n}\Big) L_{n-l-1}^{(2l+1)}\Big(\frac{2r}{n}\Big) \quad . \tag{10}$$

In this expression, $L_n^{(\alpha)}(x)$ are the Sonine-Laguerre polynomials, rather than the more restricted Laguerre polynomials (for which α must be integer). This distinction is important in subsequent sections. The Sonine-Laguerre polynomials are defined by

$$L_n^{(\alpha)}(x) = \sum_{p=0}^{n} (-x)^p \frac{\Gamma(n+\alpha+1)}{p!\,(n-p)!\,\Gamma(p+\alpha+1)} \quad . \tag{11}$$

With l fixed, the terms in brackets in Eq. (8) can be reinterpreted in terms of the hamiltonian h_+ of Eq. (6) as $h_+ - \epsilon_n$. The requirement that the ground-state energy be zero permits the separation of h_+ and ϵ_n, so that the supersymmetric partner hamiltonian h_- and the supersymmetry generator Q can be found [3]. These can be given by specifying the function U of Eq. (5), which is

$$U(y) = \frac{y}{l+1} - 2(l+1)\ln y \quad . \tag{12}$$

Explicitly, the hamiltonian h_- looks like h_+ but with the constant $l(l+1)$ replaced with $(l+1)(l+2)$, i.e.,

$$h_- - h_+ = \frac{(2l+1)}{y^2} \quad . \tag{13}$$

This shift implies that for each l the eigenfunctions of h_- are $R_{n,l+1}$, where $n \geq 2$. Together with the continuum states, they form a complete and orthonormal set.

This formalism can be given a useful physical interpretation as follows. Consider the case $l = 0$. The bosonic sector then describes the s orbitals of hydrogen. Since the spectrum of h_- is degenerate with that of h_+ except for the ground state, and since the supersymmetry generator Q acts on the radial part of the hydrogen wavefunctions but leaves the spherical harmonics untouched, h_- describes a physical system that appears hydrogenic but that has the 1s orbitals inaccessible. One way of realizing this in practice is to fill the 1s orbitals with electrons, thereby excluding the valence electron by the Pauli principle. The element with filled 1s orbitals and one valence electron is lithium. This suggests that h_- should be interpreted as an effective one-body hamiltonian describing the valence electron of lithium when it occupies the s orbitals. At this level, the description cannot be exact because most of the electron-electron interactions are disregarded. However, these can be introduced as supersymmetry-breaking terms. One procedure for this is outlined in a later section of this paper. Even in the absence of such terms, some experimental support for this atomic supersymmetry can be adduced; see ref. [3].

By redefining the energy of the 2s orbital in lithium to be zero, the hamiltonian h_- becomes a suitable choice for a bosonic hamiltonian of a second supersymmetric quantum mechanics. The fermionic partner can be constructed, and an analogous interpretation to the one above can be made. This suggests the s orbitals of lithium and sodium should also be viewed as supersymmetric partners. The process can be repeated for s orbitals and can also be applied for other values of the angular quantum number l, leading to supersymmetric connections among atoms and ions across the periodic table. In the exact-symmetry limit, these connections all involve integer shifts in l and are linked to the Pauli principle. See ref. [3] for more details.

OSCILLATOR REFORMULATION: EXACT CASE

Before describing a method for incorporating supersymmetry-breaking effects, it is appropriate to discuss an alternative formulation of the exact-symmetry case using harmonic oscillators. This section outlines the connections that exist between the radial equations for atomic supersymmetry generalized to arbitrary dimensions and those for the supersymmetric harmonic oscillator [4].

Consider first the d-dimensional Coulomb problem. Upon separation into an angular and a radial part, the radial equation appears:

$$\Big[-\frac{d^2}{dy^2} - \frac{1}{y} + \frac{(l+\gamma)(l+\gamma+1)}{y^2} - \frac{1}{2}E_{dn}\Big]v_{dnl}(y) = 0 \quad . \tag{14}$$

Here, atomic units are used, and

$$y = 2r \quad , \qquad E_{dn} = -\frac{1}{2(n+\gamma)^2} \quad , \qquad \gamma = \tfrac{1}{2}(d-3) \quad . \tag{15}$$

The radial wave functions are given by

$$v_{dnl}(y) = c_{dnl}\; y^{l+\gamma+1} \exp\big(-y/2(n+\gamma)\big) L_{n-l-1}^{(2l+2\gamma+1)}\big(y/(n+\gamma)\big) \quad , \tag{16}$$

where c_{dnl} is a normalization constant.

As before, this one-variable equation can play the role of the bosonic hamiltonaian in a supersymmetric quantum mechanics. Appropriately redefining the energy zero so that the ground state has vanishing eigenvalue permits the identification of h_+ and hence of U, Q and h_-. For example,

$$U(y) = \frac{y}{l+\gamma+1} - 2(l+\gamma+1)\ln y \quad . \tag{17}$$

Just as for the $d = 3$ case, the fermionic and bosonic hamiltonians differ only by the replacement of l by $l+1$, so that

$$h_- - h_+ = \frac{2(l+\gamma+1)}{y^2} \quad . \tag{18}$$

As required, all the results of the previous section are recovered if $d = 3$, i.e., $\gamma = 0$.

Consider next the supersymmetric harmonic oscillator. For convenience, analogous variables to the Coulomb-problem quantities y, d, n, l, $h_\pm$ are now denoted by upper case symbols Y, D, N, L, $H_\pm$. Thus, upon separation of the angular and radial parts of the Schrödinger equation for the D-dimensional harmonic oscillator, the radial equation is obtained as:

$$\Big[-\frac{d^2}{dY^2} + Y^2 + \frac{(L+\Gamma)(L+\Gamma+1)}{Y^2} - 2E_{DN}\Big]V_{DNL}(Y) = 0 \quad . \tag{19}$$

Here, atomic units have again been used for simplicity, and the oscillator is assumed to have unit frequency. The radial variable is now Y, and

$$E_{DN} = \tfrac{1}{2}(2N + 2\Gamma + 3) \quad , \qquad \Gamma = (D-3)/2 \quad . \tag{20}$$

The radial wave functions are given by

$$V_{DNL}(Y) = C_{DNL}\ Y^{L+\Gamma+1} \exp(-Y^2/2) L^{(L+\Gamma+1/2)}_{N/2-L/2}(Y^2) \quad , \tag{21}$$

with C_{DNL} a normalization constant. Note that the usual expressions for the harmonic oscillator in three dimensions are recovered when $\Gamma = 0$.

If Eq. (19) is used to define the hamiltonian H_+ of a supersymmetric quantum mechanics (a redefinition of the energy zero is again needed), then the supersymmetry is specified by a function U given by

$$U(Y) = Y^2 - 2(L + \Gamma + 1) \ln Y \quad . \tag{22}$$

It then follows that H_- differs from H_+ by the replacement of L with $L+1$. Therefore,

$$H_- - H_+ = \frac{2(L + \Gamma + 1)}{Y^2} \quad . \tag{23}$$

So far, four eigenspectra associated with the supersymmetric Coulomb and oscillator problems have been introduced, defined by the four hamiltonians h_+, h_-, H_+, and H_-. The hamiltonians h_+ and h_- are related by the map $l \rightarrow l + 1$, and the hamiltonians H_+ and H_- are related by $L \rightarrow L + 1$. The next step is to relate the d-dimensional Coulomb problem to the D-dimensional oscillator, i.e., connect h_+ to H_+.

It can be shown [4] that an eigenfunction of h_+ can be transformed by a one-parameter mapping into an eigenfunction of H_+. Explicitly, the functions v_{dnl} and V_{DNL} are connected by the equation

$$v_{dnl}\big((n + \gamma)Y^2\big) = K_{DNL} Y^{1/2} V_{DNL}(Y) \quad , \tag{24}$$

where K_{DNL} is a proportionality constant and

$$D = 2d - 2 - 2\lambda \quad , \qquad N = 2n - 2 + \lambda \quad , \qquad L = 2l + \lambda \quad . \tag{25}$$

The integer λ is the mapping parameter. Notice in particular that only oscillators in even dimensions appear. Note also that if it is desired that more than one eigenfunction of h_+ be mapped into the eigenspace of a specified H_+, then the possible choices of D, N, L, and λ can become constrained.

The existence of the one-parameter map between h_+ and H_+ combined with the supersymmetry maps evidently establishes connections between any two of the four hamiltonians h_+, h_-, H_+, and H_-. More details can be found in ref. [4].

BROKEN SUPERSYMMETRY AND QUANTUM-DEFECT THEORY

As noted above, atomic supersymmetry in the exact limit is not physically realized because the valence electron interacts with the core electrons by more than the Pauli principle. This section discusses the incorporation of supersymmetry-breaking effects in the context of alkali-metal atoms.

One important effect of the interactions between the valence electron and the core is the change in energy eigenvalues relative to the hydrogenic case. In alkali-metal

atoms, the Rydberg series [5] provides a simple formula for the measured energies, given by

$$E_{n^*} = -\frac{1}{2n^{*2}} \quad . \tag{26}$$

In this expression,

$$n^* = n - \delta(n,l) \quad , \tag{27}$$

where $\delta(n,l)$ is called the quantum defect. For a fixed value of l and increasing n, it turns out that the quantum defects rapidly attain asymptotic values: $\delta(n,l) \simeq \delta(l)$.

The changes in the energy eigenvalues imply that the exact atomic supersymmetry is broken. The breaking can be viewed as an additional contribution H_B to the supersymmetric hamiltonian H_S of Eq. (4). For example, if h_+ arises from the radial equation for hydrogen and h_- is interpreted as the radial equation for the valence electron of lithium in the exact-supersymmetry limit, then the hamiltonian H describing the two systems *including* supersymmetry-breaking effects can be taken as

$$H = H_S + H_B \quad , \tag{28}$$

where H_B has the form

$$H_B = \begin{pmatrix} 0 & 0 \\ 0 & V_B(y) \end{pmatrix} \tag{29}$$

and $V_B(y)$ is such as to generate the observed energy eigenspectrum of lithium.

The determination of a suitable V_B is not straightforward. However, it turns out that a functional form for V_B can be found that yields analytical eigenfunctions as solutions to the Schrödinger equation [6]. It is

$$V_B(y) = \frac{l^*(l^*+1) - l(l+1)}{y^2} + \frac{n^2 - n^{*2}}{4n^2 n^{*2}} \quad . \tag{30}$$

Here, l^* is a modified angular quantum number given by

$$l^* = l + i(l) - \delta(l) \quad , \tag{31}$$

where $i(l)$ is an integer parameter shifting the angular quantum number in a manner characteristic of supersymmetry. (If desired, $\delta(l)$ could be replaced by $\delta(n,l)$.) This model effectively replaces the hydrogenic radial equation with one of similar form but involving n^* and l^* rather than n and l.

By construction, the energy eigenvalues are those of the physical atom. The resulting eigenfunctions $R^*_{n^*l^*}(r)$ are analytical and are given by

$$R^*_{n^*l^*}(r) = \frac{2}{n^{*2}} \Big[\frac{\Gamma(n^*-l^*)}{\Gamma(n^*+l^*+1)}\Big]^{\frac{1}{2}} \Big(\frac{2r}{n^*}\Big)^{l^*} \exp\Big(-\frac{r}{n^*}\Big) L^{(2l^*+1)}_{n-l-i-1}\Big(\frac{2r}{n^*}\Big) \quad . \tag{32}$$

The Sonine-Laguerre polynomials enter again because

$$n^* - l^* - 1 \; = n - l - i(l) - 1 \tag{33}$$

remains integer. For asymptotic quantum defects $\delta(l)$ and including the continuum states, these eigenfunctions form an orthonormal and complete set.

More details about this construction can be found in ref. [6]. (A connection to parastatistics is elucidated in ref. [7].) There is a reasonable body of evidence to

support the notion that the analytical eigenfunctions provide a good model for the valence electron, especially in alkali-metal atoms. For instance, transition probabilities calculated with the analytical eigenfunctions agree with experiment and with accepted values [8]. (Some recursion formulae for matrix elements are given in ref. [9].) Transition probabilities for other elements, notably alkaline-earth ions, have also been obtained in this way [10]. Moreover, these analytical eigenfunctions have been used as trial wavefunctions in detailed atomic calculations [11]. Stark maps for the alkali-metal atoms can also be calculated using the model [12]. The resulting clear anticrossings and small-field quadratic Stark effects for the s and p orbitals are in agreement with experiment. For example, the model yields Stark maps for the $n = 15$ lines of lithium and sodium that are indistinguishable from the numerical and experimental results of ref. [13]. (Ref. [12] also studied other possible quantum-mechanical supersymmetries involving hydrogen. In particular, a double sqm(2) appears when the separation is carried out in parabolic coordinates.) The model is expected to break down at short distances from the nucleus (of order of the core size), but despite this some dominant features of the fine structure in alkali-metal atoms are correctly reproduced and the Landé semiempirical formula naturally appears [14].

OSCILLATOR REFORMULATION: BROKEN CASE

This section presents a reformulation of the analytical quantum-defect model in terms of oscillators with radial equation modified by an anharmonic term. For generality, the connection between the two is treated in arbitrary dimensions. The oscillator models that appear have analytical solutions. The link between the two theories is via a three-parameter map. In the limit of vanishing quantum defect, this map provides a generalization of the ones of ref. [4] discussed above. For example, it can be used to connect a Coulomb problem to an anharmonic oscillator with *odd* dimensionality.

The first step is to construct the d-dimensional extension of the quantum-defect model of ref. [6]. This is done by adding to the hamiltonian (14) an extra term $V_B^d(y)$ generalizing V_B in Eq. (30), with

$$V_B^d(y) = \frac{(l^*+\gamma)(l^*+\gamma+1) - (l+\gamma)(l+\gamma+1)}{y^2} + \frac{(n+\gamma)^2 - (n^*+\gamma)^2}{4(n+\gamma)^2(n^*+\gamma)^2} \quad . \tag{34}$$

Here, n^* and l^* are modified quantum numbers given by

$$n^* = n - \delta(d,l) \quad , \qquad l^* = l + i(d,l) - \delta(d,l) \quad , \tag{35}$$

in analogy with Eqs. (27) and (31), with

$$\gamma = \tfrac{1}{2}(d-3) \tag{36}$$

as before. The ensuing radial equation in atomic units is

$$\Big[-\frac{d^2}{dy^2} - \frac{1}{y} + \frac{(l^*+\gamma)(l^*+\gamma+1)}{y^2} - \frac{1}{2}E_{dn^*}\Big] v^*_{dn^*l^*}(y) = 0 \quad , \tag{37}$$

where

$$E_{dn^*} = -\frac{1}{2(n^*+\gamma)^2} \quad . \tag{38}$$

The radial wave functions solving Eq. (37) are given by

$$v^*_{dn^*l^*}(y) = c^*_{dn^*l^*}\ y^{l^*+\gamma+1}\exp\bigl(-y/2(n^*+\gamma)\bigr)L^{(2l^*+2\gamma+1)}_{n-l-i-1}\bigl(y/(n^*+\gamma)\bigr)\quad , \tag{39}$$

where $c^*_{dn^*l^*}$ is a normalization constant.

To identify a mapping between the quantum-defect theory and an oscillator-type model, a (supersymmetry-breaking) term $V^D_B(Y)$ modifying the harmonic-oscillator radial hamiltonian (19) is needed. A suitable choice is

$$V^D_B(Y) = \frac{(L^*+\Gamma)(L^*+\Gamma+1)-(L+\Gamma)(L+\Gamma+1)}{Y^2} + 2(N-N^*)\quad . \tag{40}$$

In this expression,

$$\Gamma = \tfrac{1}{2}(D-3) \tag{41}$$

as before, and the modified quantum numbers N^* and L^* are given by

$$N^* = N - 2\Delta(D,N,L)\quad ,\qquad L^* = L + 2I(D,L) - 2\Delta(D,N,L)\quad , \tag{42}$$

where the integer $2I(D,L)$ is a supersymmetry-type shift and 2Δ represents a quantum anharmonicity (which can be viewed as an oscillator 'defect'). The factors of two are introduced for notational simplicity in what follows. The extra term (40) introduces an anharmonic piece into the oscillator potential, which in turn changes the energy eigenspectrum. In atomic units with a unit-frequency oscillator, the anharmonic radial equation becomes

$$\Bigl[-\frac{d^2}{dY^2} + Y^2 + \frac{(L^*+\Gamma)(L^*+\Gamma+1)}{Y^2} - 2E_{DN^*}\Bigr]V^*_{DN^*L^*}(Y) = 0\quad , \tag{43}$$

where the energy eigenvalues are shifted according to

$$E_{DN^*} = \tfrac{1}{2}(2N^*+2\Gamma+3)\quad . \tag{44}$$

The eigensolutions for this anharmonic oscillator are

$$V^*_{DN^*L^*}(Y) = C^*_{DN^*L^*}\ Y^{L^*+\Gamma+1}\exp\bigl(-Y^2/2\bigr)L^{(L^*+\Gamma+1/2)}_{N/2-L/2-I}\bigl(Y^2\bigr)\quad , \tag{45}$$

where $C^*_{DN^*L^*}$ is a normalization constant.

The map between the radial equations for the d-dimensional quantum-defect theory and the D-dimensional anharmonic oscillator connects the eigensolutions $v^*_{dn^*l^*}$ and $V^*_{DN^*L^*}$. It is given by

$$v^*_{dn^*l^*}\bigl((n^*+\gamma)Y^2\bigr) = K^*_{DN^*L^*}Y^{1/2}V^*_{DN^*L^*}(Y)\quad , \tag{46}$$

where $K^*_{DN^*L^*}$ is a proportionality constant and

$$D = 2d-2-2\lambda\ ,\quad N = 2n+2(\Delta-\delta)-2+\lambda\ ,\quad L = 2l+2(\Delta-\delta)-2(I-i)+\lambda\ . \tag{47}$$

For fixed d, n, l, δ, and i there are three quantities that effectively act as mapping parameters: λ, Δ, and I. For the eigenfunctions (45) to exist I must be integer, so the supersymmetry-type shift $2I$ in L is an even integer. Since D must also be an integer, λ is integer or half-integer. The quantum numbers N and L are also integer,

which implies that $2(\Delta - \delta) + \lambda$ must be integer. Note that this generalizes the exact-symmetry case: the requirement that λ be a whole integer is no longer needed because half-integral values can be absorbed in the difference $2(\Delta - \delta)$. This means, for example, that when the Coulomb problem is treated in the exact limit ($\delta = 0$) it is now possible to map it into a (modified) oscillator in an *odd* number of dimensions, provided Δ is quarter-integer valued. As in the exact case, for maps between *sets* of states with specified d and D, further restrictions on the possible values of D, N, L, I, Δ and λ may appear.

SUPERSYMMETRIC OSCILLATORS AND THE PENNING TRAP

In this section, I demonstrate that geonium atoms provide a physical realization of a $D > 1$ supersymmetric harmonic oscillator. For simplicity, the specific case $D = 2$ is considered, although under suitable conditions higher values of D may appear. In practical situations the supersymmetry is broken for reasons to be described. The analytical anharmonic oscillator model introduced in the previous section should provide a good approximation to the exact wavefunctions for this case. Space limitations prevent more than a sketch of the relevant physics being given here; details will appear elsewhere.

Geonium atoms are formed by a set of charged particles bound in a Penning trap [15], which is a suitable combination of a homogeneous magnetic field and an electrostatic quadrupole potential. The simplest geonium atom has just one trapped particle of charge e and mass m [16]. Successively adding further electrons in the trap generates elements of the geonium periodic table.

For simplicity, consider the idealized Penning trap with electromagnetic fields specified in cylindrical coordinates (ρ, θ, z) by

$$\mathbf{B} = B\hat{z} \quad , \qquad \phi = \tfrac{1}{2}\frac{V}{d^2}(z^2 - \tfrac{1}{2}\rho^2) \quad . \tag{48}$$

The quantity d is a measure of the trap dimension and is to be specified in terms of the configuration of the quadrupole electrodes. For (stable) trapping, $eV > 0$. The quantum-mechanical motion of the particle in the field $\mathbf{B}$ is that of a harmonic oscillator with binding frequency equal to the cyclotron frequency, given in SI units by

$$\omega_c = \frac{|eB|}{m} \quad . \tag{49}$$

(In fact, there are *two* oscillators involved in this motion, but only one enters the quantum hamiltonian.) Similarly, the electrostatic field generates an axial harmonic motion independent of the cyclotron motion, with axial frequency

$$\omega_z = \sqrt{\frac{eV}{md^2}} \quad . \tag{50}$$

These are the motions of primary interest here.

The simultaneous presence of electric and magnetic fields also generates another (unbound) circular motion, called the magnetron motion, with frequency ω_m. For simplicity, this is largely disregarded here. The eigenvalue spectrum of the system is split by all these interactions and also (for particles with spin $\mathbf{S}$) by the spin interaction $-\mathbf{S} \cdot \mathbf{B}$. The latter splitting implies the existence of a supersymmetry of

the type discussed in refs. [17]. This supersymmetry is not directly relevant to the discussion here, and the spin degree of freedom is neglected in what follows.

The combination of the cyclotron and axial motions forms a system of two one-dimensional oscillators. This becomes a physical realization of a $D = 2$ harmonic oscillator when the applied electromagnetic fields are chosen such that $\omega_c = \omega_z$, i.e.,

$$mV = |e|B^2 d^2 \quad . \tag{51}$$

The quantum problem can then be separated in polar coordinates and the radial equation has the general form of Eq. (19) with $\Gamma = -\frac{1}{2}$, and with suitable constant factors inserted to allow for non-unit binding frequency and for SI units.

For fixed L, this oscillator can be used as the bosonic partner H_+ in a supersymmetric quantum mechanics. The partner hamiltonian H_- is specified by Eq. (23). It represents a system having an eigenspectrum degenerate with the bosonic sector but with the ground state missing. As in atomic supersymmetry, one practical realization of this is to fill the ground state with particles and invoke the Pauli principle. If $L = 0$, for example, H_+ describes the S orbitals of the simplest geonium atom with one trapped particle. Then, H_- can be interpreted as an effective theory describing the behavior of the 'valence' particle in the S orbitals of a more complex geonium atom in which the 1S (and 2P) orbitals are filled. This interpretation invokes the approximation in which particle interactions other than those implied by the Pauli principle are disregarded. All the atomic supersymmetries of ref. [3] have analogues in this system. For example, there are connections between pairs of geonium atoms throughout the geonium periodic table.

There exists a mapping between the $d = 3$ Coulomb problem and the $D = 2$ harmonic oscillator, as discussed above. The connection between the eigenfunctions given in Eq. (24) therefore establishes a correspondence between eigenfunctions of elements in the usual periodic table and the geonium periodic table. The map (24) is fixed here by setting $\lambda = 1$, so that

$$N = 2n - 1 \quad , \qquad L = 2l + 1 \quad . \tag{52}$$

Moreover, in the exact supersymmetry limit this map induces other maps involving the supersymmetric partners so that all four hamiltonians are linked.

The supersymmetries are broken by the interaction of the valence particle with the 'core' of the geonium atom. These interactions will shift the eigenenergy of the valence particle from $E_N = N + 1$ (in the level with quantum numbers N, L) to some other energy

$$E_{N^*} = N^* + 1 \quad , \tag{53}$$

where by definition

$$N^* = N - 2\Delta(N, L) \quad . \tag{54}$$

A model incorporating these exact new eigenenergies and yielding analytical solutions has been introduced in the previous section. In the present case, it is obtained by adding an anharmonic term to the oscillator hamiltonian, giving the radial equation (43) with $\Gamma = -\frac{1}{2}$ and with suitable dimension-correcting factors inserted. The analytical solutions are given by a corresponding modification of Eq. (45).

It is physically plausible to conjecture that the quantum anharmonicity rapidly approaches an asymptotic value as N becomes large, i.e.,

$$\Delta(N, L) \simeq \Delta(L) \quad . \tag{55}$$

In this case, the eigensolutions form a complete and orthogonal set. The previous section also provides a mapping between this theory and the analytical quantum-defect theory for ordinary atoms. Combined with the magnetron and spin splittings, the anharmonic model is likely to provide a simple method for calculations of physical properties of the valence particle in geonium atoms.

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