1. Properties of Spin-1/2 relativistic particles

Lorentz transformations. Under Lorentz transformations, \( x \to x' = \Lambda x \) the free Dirac field, \( \psi_\alpha(x) \) transforms according to,

\[
\psi_\alpha(x) \to \psi'(x')_\alpha = S(\Lambda)_{\alpha \beta} \psi_\beta(x)
\]

(1)

Where the 4 \times 4 Dirac representation, \( S(\Lambda) \) is given by

\[
S(\Lambda) = e^{-i/2[\gamma^\mu, \gamma^\nu] \omega_{\mu\nu}}
\]

(2)

Here, \( \gamma^\mu = i/2[\gamma^\mu, \gamma^\nu] \) and \( \omega_{\mu\nu} \) are the generators of the Lorentz transformation, \( \Lambda = \Lambda(\omega) \). Assume \( \Lambda(\omega) \) is a pure Lorentz boost specified by two momentum 4-vectors; let \( p^\mu = (E = \sqrt{m^2 + \mathbf{p} \cdot \mathbf{p}}, \mathbf{p}) \) and \( p'^\mu = (E' = \sqrt{m^2 + \mathbf{p}' \cdot \mathbf{p}'}, \mathbf{p}') \)

4-vectors specify \( \omega \) by, \( \omega = \Lambda^\dagger(\omega)p_\mu \).

1.1 (15pt) Show that the corresponding matrix \( S \) is given by

\[
S = \cosh \frac{|\omega|}{2} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \sinh \frac{|\omega|}{2} \begin{pmatrix} 0 & \sigma^\cdot \omega \\ \sigma^\cdot \omega & 0 \end{pmatrix}
\]

(3)

where

\[
\frac{\sigma^\cdot \omega}{|\mathbf{p}' - \mathbf{p}|}
\]

and \( |\omega| \) is given by

\[
\cosh |\omega| = \frac{(E' + E)^2 + (\mathbf{p}' + \mathbf{p})^2}{(E' + E)^2 - (\mathbf{p}' + \mathbf{p})^2}
\]

(4)

(5)

2 The quantum, free field \( \psi_\alpha(x) \) can be decomposed as

\[
\psi_\alpha(x) = \sum_{\lambda = \pm} \int \frac{d^3p}{(2\pi)^3 2E} \left[ u(\mathbf{p}, \lambda) h(\mathbf{p}, \lambda) e^{-i\mathbf{p} \cdot \mathbf{x}} + v(\mathbf{p}, \lambda) d^\dagger(\mathbf{p}, \lambda) e^{i\mathbf{p} \cdot \mathbf{x}} \right]
\]

(6)

with \( p^0 = E = \sqrt{m^2 + \mathbf{p}^2} \), and \( N = u \) and \( N = v \) being the positive and negative energy eigenvector of the Dirac operator,

\[
(\tilde{p}^\mu \gamma^\mu - m) N = 0
\]

(7)

with \( \tilde{p}^\mu = (\pm p^0, \mathbf{p}) \) for \( u \), (+) and \( v \) (–) respectively.
2.1 (5pt) Show that, \( d^3p / 2E = d^3p' / 2E' \) i.e. measure is invariant under \( \Lambda \).

2.2 (10pt) Define single particle state by
\[
|p, \lambda, p\rangle \equiv b^\dagger(p, \lambda)|0\rangle
\]
and single anti-particle state by
\[
|p, \lambda, a\rangle \equiv d^\dagger(p, \lambda)|0\rangle
\]
were \( |0\rangle \) satisfies,
\[
b(p, \lambda)|0\rangle = d(p, \lambda)|0\rangle = 0
\]
By imposing the relativistic normalization condition,
\[
\langle p', \lambda', p|p, \lambda', p \rangle = (2\pi)^4 2E\delta_{\lambda'} \delta^{\lambda}(p' - p)
\]
Calculate the normalization of the spinors \( u \) and \( \psi \).

2.3 (20pt) For the transformed field we have, one has
\[
\psi'(x') = \sum_{\lambda, \lambda'} \int \frac{d^3p'}{(2\pi)^3 2E'} \left[ u(p', \lambda') b(p', \lambda') e^{-i\vec{p}\cdot\vec{x'}} + v(p', \lambda') d^\dagger(p', \lambda') e^{i\vec{p}\cdot\vec{x'}} \right]
\]
Starting from the relation between \( \psi'(x') \) and \( \psi(x) \) (Eq. 1), show that
\[
|p', \lambda p\rangle = \sum_{\lambda} |p, \lambda, p\rangle D(\Lambda)^{-1}_{\lambda\lambda'}
\]
where \( D \) is the Wigner rotation (as discussed in class),
\[
D(\Lambda) = \frac{(E' + m)(E + m) + \vec{p}\cdot\vec{p}' + i[\vec{p}\times\vec{p}']\cdot\vec{\sigma}}{\sqrt{2(E' + m)(E + m)(E' E + m^2 + \vec{p}'\cdot\vec{p})}}
\]

3. Charges Define the charge operator by \( Q \)
\[
Q = \int d^3x \psi^\dagger(x)\psi(x)
\]

3.1 (5pt) Show that \([H, Q] = 0\) where \( H \) is the free Dirac Hamiltonian
\[
H = \int d^3x \psi^\dagger(x) \left[ i\vec{\alpha} \cdot \vec{\nabla} + \beta m \right] \psi(x)
\]

3.2 (5pt) Show that the single particle and antiparticle states defined in 2 above are eigenstates of \( Q \) with eigenvalue +1 and -1 respectively.