Electromagnetic structure of the Nucleon in the quark model.

In the quark model one assumes that nucleon (proton or neutron) is a bound state of 3 quarks. Assume one can then write the nucleon state \([\mathbf{P}_N, \lambda, N]\) of a nucleon \(N = 1/2, (proton)\), \(-1/2(neutron)\) of momentum \(\mathbf{P}_N\) and spin projection \(\lambda_N = \pm 1/2\) as, (schematically)

\[
|\mathbf{P}_N, \lambda_N, N\rangle = \sum_{abc} b^\dagger a b^\dagger b |0\rangle \Psi^N(abc)
\]

where \(a, b, c\) stand for a complete set of single quark quantum numbers each, \(i.e.\ a = (k, \lambda, i, \epsilon), \ k - \text{quark’s momentum}, \ \lambda = \pm 1/2\ \text{spin projection}, i = +1/2(up), -1/2(down)\ - \text{quark’s flavor}, c = 1, 2, 3 - \text{quark’s color}.\) Assume that the wave function \(\Psi^N\) is given by

\[
\Psi^N(abc) = A(2\pi)^3 \delta^3(\mathbf{P}_N - \mathbf{k}_a - \mathbf{k}_b - \mathbf{k}_c) \frac{1}{\sqrt{6}} \epsilon_{\alpha a, c_1, \epsilon_c} \Phi(k_a, k_b, k_c)
\]

\[
\times \frac{1}{\sqrt{2}} \left[ \phi_{MS}(\lambda_a, \lambda_b, \lambda_c) \chi_{MS}^N(i_a, i_b, i_c) + \phi_{MA}(\lambda_a, \lambda_b, \lambda_c) \chi_{MA}^N(i_a, i_b, i_c) \right]
\]

(2)

Here \(\Psi\) is an orbital, spin-flavor independent wave function, same for proton and neutron. Assume that this wave function is the ground state (momentum space) solution of a 3-particle harmonic oscillator,

\[
\frac{1}{2} \left[ \sum_{i=1}^{3} \frac{\mathbf{k}_i^2}{2m} + m\omega^2 \sum_{i>j} (x_i - x_j)^2 \right] \Phi(k_1, k_2, k_3) = E_0 \Phi(k_1, k_2, k_3)
\]

(3)

The spin wave functions, \(\phi_{MS}(\lambda_i)\) (\(MS = \text{mixed symmetric}\)) and \(\phi_{MA}(\lambda_i)\) (\(MA = \text{mixed antisymmetric}\)) are obtained by coupling spins of quarks one and two \((S_1 = 1/2)\) to \(S_{12} = 1\) and \(S_{12} = 0\) respectively and then coupling the third quark to an overall spin \(S_N = 1/2\) nucleon,

\[
\phi_{MS}(\lambda_i) = \sum_{\lambda_{12}} \langle \lambda_{12} \frac{1}{2}, \lambda_1; \frac{1}{2}, \lambda_2 | 1, \lambda_{12} \rangle \langle 1, \lambda_{12} \frac{1}{2}, \lambda_3 | \frac{1}{2}, \lambda_N \rangle
\]

(4)

and

\[
\phi_{MA}(\lambda_i) = \langle \lambda_1 \frac{1}{2}, \lambda_2; \frac{1}{2}, \lambda_3 | 0, 0 \rangle \langle 0, 0 \frac{1}{2}, \lambda_3 | \frac{1}{2}, \lambda_N \rangle
\]

(5)

and \(\langle j_1, m_1; j_2, m_2 | j, m \rangle\) is the CG coefficient.
The flavor wave function $\chi_{MS}(i_i)$ and $\chi_{MA}(i_i)$ are just like spin wave functions $\phi_{MS}$ and $\phi_{MA}$ if you replace $\lambda = +1/2$ by $i = +1/2$ (up) and $\lambda = -1/2$ by $i = -1/2$ (down) and $\Lambda_N = +1/2$ by $N = +1/2$ (proton) and $\Lambda_N = -1/2$ by $N = -1/2$ (neutron).

The color wave function, $\epsilon_{c_1,c_2,c_3}$ is the antisymmetric Levi-Civita tensor $\epsilon_{123} = 1$.

(10pt) Using a relativistic normalization condition, in the nonrelativistic limit, for the the nucleon and quark states

$$\langle \mathbf{P}_N', \lambda_N', N|R_1(0)|\mathbf{P}_N, \lambda_N, N \rangle = (2\pi)^3 2E_N(\mathbf{P}_N')\delta^3(\mathbf{P}_N' - \mathbf{P}_N).$$  \hspace{1cm} (6)

$$\{b_{a,b}\} = \delta_{ab} = 2E_q(\mathbf{k}_a)\delta_{\lambda_a,\lambda_b}\delta_{\iota_a,\iota_b}\delta_{c_a,c_b}(2\pi)^3\delta^3(\mathbf{k}_a - \mathbf{k}_b).$$  \hspace{1cm} (7)

Derive the normalization condition of the orbital wave function $\Phi(\mathbf{k}_i)$. In the nonrelativistic limit, $E_N = \sqrt{M^2 + \mathbf{P}_N^2} \to M_N (M_N = nucleon mass)$, and $E_q = \sqrt{m^2 + \mathbf{k}^2} \to m$ ($m$ = quark mass, assume the same for up and down quarks).

(10pt) Solve the harmonic oscillator of the ground state wave function and normalize it according to the above.

Now we want to calculate the matrix element of the e.m. current in the nucleon. The full expression for the current matrix element is given by

$$\langle \mathbf{P}_N', \lambda_N', N|R_\mu(0)|\mathbf{P}_N, \lambda_N, N \rangle$$
$$= \bar{u}_N(\mathbf{P}_N', \lambda_N') \left[ F_1^N(Q^2)\gamma^\mu + \frac{k}{2M} F_2^N(Q^2)i\sigma^\mu\nu q_\nu \right] u(\mathbf{P}_N, \lambda_N)$$  \hspace{1cm} (8)

with $Q^2 = -q^2 = (P_N' - P_N)^2 = (E_N(\mathbf{P}_N') - E_N(\mathbf{P}_N))^2 - (\mathbf{P}_N' - \mathbf{P}_N)^2$. The spinors are given by

$$u_N(\mathbf{P}_N, \lambda_N) = \frac{1}{\sqrt(E_N + M)} \left( \begin{array}{c} (E + M)\chi(\lambda) \\ \bar{\sigma} \cdot \mathbf{P}_N \chi(\lambda) \end{array} \right)$$  \hspace{1cm} (9)

(10pt) In the nonrelativistic limit ($E_N \to M$) calculate the RHS of Eq.(8) and express the form factors $F_1$ and $F_2$ in terms of spin-nonflip $\lambda_N = \lambda_N = +1/2$ and spin-flip $\lambda_N = -\lambda_N = +1/2$ matrix elements.

(10pt) the em current operator is expressed in terms of up and down quark fields as,

$$j^\mu(0) = \frac{2}{3} e \sum_{\alpha=1}^3 \bar{\psi}_{u,\alpha}(0)\gamma^\mu \psi_{u,\alpha}(0) - \frac{1}{3} \sum_{\beta=1}^3 \bar{\psi}_{d,\beta}(0)\gamma^\mu \psi_{d,\beta}(0)$$  \hspace{1cm} (10)
Here $c = 1, 2, 3$ is the quark’s color and the em current is the same for each color type ($\sum_c$) The fields are given by

$$\psi_{\lambda,c}(0) = \sum_{\lambda} \int \frac{d^3k}{2E_q(2\pi)^3} \left[ u(k, \lambda) b(k, \lambda, i, c) + v(-k, \lambda, i, c) d\dagger(k, \lambda, i, c) \right]$$  \hspace{1cm} (11)$$

The spinors are just like in Eq. (9) with a replacement of nucleon parameters by quark parameters.

(10pt) Explain why in this model only the quark operators ($b$ and $b\dagger$) need to be retained and calculate $j^0(0)$ and $\tilde{j}(0)$ in the nonrelativistic limit in terms of these operators.

(60pt) Calculate $F_1$ and $F_2$ in this model. Show that the ratio of neutron to proton magnetic moment is given by $\mu_n/\mu_p = -2/3$ and plot the f.factors as a function of $Q^2$. 