momentum \( j_3 \) of particle 3 gives a total angular momentum of zero, we must have \( j = j_3, m = -m_3 \). According to (106.2) we can then write

\[
\psi_0 = \frac{1}{\sqrt{(2j+1)}} \sum_m (-1)^j-m \psi_m^{(2)} j_3 m_3. \tag{106.7}
\]

This formula is to be compared with (106.4) (in which we replace \( j_3, m_3 \) by \( j, m \)). Here, however, we must first take into account the fact that the rule for constructing the sum in (106.7) in accordance to (106.3) does not correspond to the rule for constructing the sum in (106.4). To bring (106.7) to the form (106.4) we must, as is easily seen, interchange pairs of upper and lower indices corresponding to particles 1 and 3. This leads to an additional factor \((-1)^{j_1-j_3} j \). The result is

\[
\psi_{jm} = (-1)^{j_1-j_3-m} \sqrt{(2j+1)} \sum_{m_1, m_2} \binom{j_1 \ j_2 \ j}{m_1 \ m_2 \ -m} \psi_{m_1 m_2}^{(2) j_3 m_3}, \tag{106.8}
\]

where the summation over \( m_1 \) and \( m_2 \) is subject to the condition \( m_1 + m_2 = m \).

Formula (106.8) gives the required expression for obtaining the wave function of a system from those of its two particles, which have definite angular momenta \( j_1 \) and \( j_2 \). It can be written in the form

\[
\psi_{jm} = \sum_{m_1, m_2} \langle m_1 m_2 | jm \rangle \psi_{m_1 m_2}^{(2) j_3 m_3} \quad (m_2 = m - m_1). \tag{106.9}
\]

The coefficients

\[
\langle m_1 m_2 | jm \rangle = (-1)^{j_1-j_2-m} \sqrt{(2j+1)} \binom{j_1 \ j_2 \ j}{m_1 \ m_2 \ -m} \tag{106.10}
\]

form the matrix of the transformation from the complete orthonormal set of \((2j_1 + 1)(2j_2 + 1)\) wave functions of states \( m_1 m_2 \) to the similar set of wave functions of states \( jm \) (for given values of \( j_1, j_2 \)). They are called vector addition coefficients or Clebsch–Gordan coefficients. The notation \( \langle m_1 m_2 | jm \rangle \) corresponds in the general notation for the coefficients in the expansion of one set of functions in terms of another (11.18). To simplify, we have omitted the quantum numbers \( j_1 \) and \( j_2 \), which are the same in both sets of functions. When necessary, these are included, in the form \( \langle j_1 m_1 j_2 m_2 | j_a m_a \rangle \].

* Under time reversal, the wave functions change in accordance with (50.2):

\[ \psi_{jm} \rightarrow (-1)^{j_1-j_2} \psi_{ jm} \]

It is easily verified that the function \( \psi_{jm} \) on the left of (106.8) is transformed in this way if the functions \( \psi_{jm} \) and \( \psi_{jm} \) on the right are.

* The Clebsch–Gordan coefficients are also denoted in the literature by \( C_{m_1 m_2}^{j_a} \) or \( C_{m_1 m_2,j_a}^{j_a} \).
are obtained directly from (106.14). The derivation of the formula

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} = (-1)^p \left[ \frac{(j_1 + j_2 + j_3)!(j_1 - j_3 - j_2)!(j_2 - j_1 - j_3)!(j_3 - j_1 - j_2)!}{(2p+1)!} \times \frac{1}{(j_1 + j_2)! (j_2 + j_3)! (j_3 + j_1)!} \right]^{1/2}$$

(106.18)

where $2p = j_1 + j_2 + j_3$ is even, requires a number of additional calculations; when $2p$ is odd, this $3j$-symbol is zero owing to the symmetry property (106.6).

Table 9 gives for reference the values of the $3j$-symbols for $j_3 = \frac{1}{2}$, 1, $\frac{3}{2}$, 2. For each $j_3$ the minimum number of $3j$-symbols are shown from which the remainder may be obtained by means of the relations (106.5), (106.6).
Addition of Angular Momenta

PROBLEM

Determine the angle dependence of the wave functions of a particle with spin \( \frac{1}{2} \) in states with given values of the orbital angular momentum \( l \), the total angular momentum \( J \) and component thereof \( m \).

SOLUTION. The problem is solved by the general formula (106.8), in which \( \phi^{(l)} \) must be taken as the eigenfunctions of the orbital angular momentum \( l \), the spherical harmonic functions \( Y_{lm}(\Omega) \), and \( \psi^{(J)} \) as the spin wave function \( \chi(\sigma) \) (where \( \sigma = \pm \frac{1}{2} \)):

\[
\psi_{lm} = (-1)^{l-m} \sqrt{2l+1} \sum_{\sigma=-l}^{l} \binom{i}{l-m} Y_{l,m}(\Omega) \chi(\sigma).
\]

Substituting the values of the \( \binom{i}{l-m} \) symbols, we obtain

\[
\begin{align*}
\phi_{1/2, m} &= \sqrt{\frac{2j+1}{2j}} Y_{1,m} + \sqrt{\frac{2j}{2j+1}} Y_{1,-m}, \\
\phi_{j-1/2, m} &= \sqrt{\frac{2j}{2j+1}} Y_{j-1,m} + \sqrt{\frac{2j+1}{2j}} Y_{j-1,-m}, \\
\phi_{j+1/2, m} &= \sqrt{\frac{2j}{2j+1}} Y_{j+1,m} + \sqrt{\frac{2j+1}{2j}} Y_{j+1,-m}.
\end{align*}
\]

Matrix elements of the Vector

In §106 formulae have been given for vector physical quantities in terms of their components. These formulae are in general formulae for the irreducible tensor.

The set of \( 2k+1 \) components are equivalent, as regards their spherical harmonic functions \( Y_{lm} \) (§57). This means that, by means of a similarity transformation under rotations as the \( f_{k0} \) will be denoted here by \( f_{k0} \), it is possible to find for example, \( k = 1 \) for the components of the vector by the relations

\[
f_{10} = i d_{10},
\]

cf. (57.7). The corresponding

\[
f_{20} = -\sqrt{\frac{2}{3}} d_{20}
\]

with \( a_{xx} + a_{yy} + a_{zz} = 0 \).

The construction of tensor \( T_{k0} \) is a complicated process in accordance with the formalism of tensor components, with \( k_1, k_2, k_3 \) formally expressed by these tensors. This is the reason why one form the spherical tensor expression of the formula

\[
(f_{k0} \psi_{j})_K = \sum_{\varphi, \varphi'} \langle q_1 q_2 | K \rangle
\]

which differs from the definiti