This is problem 3.11 from F& W
1. Consider a system of free spin-1/2 fermions in a static, spin-dependent external potential given by the Hamiltonian,

\[
H_{\text{ex}} = \int d^d x \hat{\psi}^\dagger_\alpha(x) V_{\alpha\beta}(x) \hat{\psi}_\beta(x)
\]  
(1)

1.1 (10pt) Use the Wick theorem to give the set of Feynman rules for the one-body Green's function in the presence of this potential.

1.2 (10pt) Show that the Dyson equation for full, interacting, the one-body Green’s function is given by

\[
G_{\alpha\beta}^{\text{ex}}(x,y) = G_{\alpha\beta}^0(x,y) + \int d^d z G_{\alpha\lambda}^0(x-z) V_{\lambda\nu}(z) G_{\nu\beta}^{\text{ex}}(z,y)
\]  
(2)

1.3 (10pt) Express the energy of the ground state in terms of the Green’s function.

This is problem 3.14 from F& W + extension.

2. Using the expression for the Green’s function in terms of the proper self energy,

\[
G_{\alpha\beta}(k) = G_{\alpha\beta}(k,\omega) = \frac{\delta_{\alpha\beta}}{\omega - \omega_k - \Sigma^*(k,\omega)}, \quad \omega_k = \frac{k^2}{2m}
\]  
(3)

2.1 (20pt) show that the energy \(\epsilon_k\) and inverse life-time, \(|\gamma_k|\) of narrow, single particle excitations are given by

\[
\epsilon_k = \omega_k + \text{Re} \Sigma^*(k,\epsilon_k)
\]  
(4)

and

\[
\gamma_k = \left[ 1 - \left. \frac{\partial \text{Re} \Sigma^*(k,\omega)}{\partial \omega} \right|_{\omega=\epsilon_k} \right]^{-1} \text{Im} \Sigma^*(k,\epsilon_k)
\]  
(5)

why the assumption of the excitation being narrow is relevant?
As an illustration consider a system of two types of spinless bosons, $i = 1, 2$, described by the following Hamiltonian

$$H = \int \frac{d^3p}{(2\pi)^3} \left( m_1 + \frac{p^2}{2m_1} \right) a_1^\dagger(p)a_1(p) + \int \frac{d^3p}{(2\pi)^3} \left( m_2 + \frac{p^2}{2m_2} \right) a_2^\dagger(p)a_2(p) + V$$

(6)

with the interaction $V$ given by

$$V = g \int \frac{d^3k}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{d^3p}{(2\pi)^3} (2\pi)^3 \delta^3(q+k-p) f(|q-k|)a_1^\dagger(k)a_2^\dagger(q)a_1(p) + c.c$$

(7)

Here $g$ is a coupling constant, and $f(x)$ is a smooth, dimensionless, function vanishing rapidly as $x \to \infty$, (e.g. $f(x) = e(-x^2/\beta^2)$, $\beta = \text{const.}$). This is a model for the interaction that describes a decay of one boson of type-1 to a pair of bosons of type-2. To allow for such a process we add masses to the kinetic energy which comes from a nonrelativistic reduction of the relativistic dispersion relation

$$E_i(p) = \sqrt{m_i^2 + p^2} \sim m_i + \frac{p^2}{2m_i}$$

(8)

As a ground state take the vacuum (no bosons of any type). Then the creation $a_1^\dagger(p)$, (annihilation, $a_i(p)$) operators, correspond to adding (removing) a boson of type-$i$ with momentum $p$.

2.2 (5pt) What are the physical dimensions of relevant quantities, $a$, $g$ in units of mass? Since this is a semi-relativistic problem we are using a system of units with $c = \hbar = 1$.

2.3 (5pt) Write down the free Green’s functions for the bosons, $G_i^0(q, \omega)$.

2.4 (20pt) Calculate $\Sigma^*$ for the boson of type 1 to the lowest nonvanishing order in the interaction.

2.5 (20pt) Using 2.1 Calculate the excitation energy of the single boson of type-1 and its lifetime, for the two cases, $m_1 > 2m_2$ and $m_1 < 2m_2$. Note that $\omega_k$ in 2.1 has to be replaced by the semi-relativistic kinetic energy we using here.

2.6 (10pt) Discuss the results for the two cases.