**Diagrammatically:**

\[ E^{\gamma,\alpha}_{\mu} = \frac{g^{\alpha}_{\mu}}{\beta^0} \left( \frac{1}{4} \right) \]

\[ \nabla \phi = \frac{\delta \phi}{\beta^0} \]

\[ \chi \phi = 0 \]

**The important part is the loop**

\[ E^{\gamma,\alpha}_{\mu} \sim 3g^4 \left( \sum \frac{f^{\beta}_{\gamma \alpha} f^{\beta}_{\gamma \alpha}}{2} \right) \int \frac{d^3h}{(2\pi)^3} \frac{(1-(h^2\beta^0)^2)}{\omega(h)|p^2-h^2|^2} \]

\[ \frac{1}{2} \left[ T^a_{\gamma \alpha} T^a_{\gamma \alpha} \right]_{oo} = \frac{1}{2} \frac{C_A}{N_c} = \frac{N_c}{2} \]

**For SU(N)**

\[ C_A = N \]

**Cosines of SU(3)**

\[ \cos^2 \theta^D = \frac{1}{2} \]

\[ \theta^D = \theta \]

**For SU(3)**

\[ I = I(\tau +) \]

\[ I = 1 \rightarrow I^2 = 2 = N \]
In QCD if $A=B$, i.e., we have 2 quarks with the same color, $T^{d}_{AA} T^{d}_{AA} > 0$ and $E_{12} > 0$ and repulsion. Also since $T^{d}$ are hermitian, $T^{d}_{AA}$ is real and so is $E_{11}$.

Now let's discuss how one would calculate higher order corrections, $E_{11}, E_{12}, ...$ etc.

For details see T.D. Lee.

\[ O(g^3) \] Here will be operators with odd # of gluons thus again $E_{12} = 0$. To $O(g^4)$ there are 2 types of contributions.

\[ E_{11} = \langle QQ | g^4 V_2 | AA \rangle \]

\[ E_{12} = \sum_{\text{inter}} \left| \langle Q\bar{Q} | g^2 V_1 | \text{inter} \rangle \right|^2 \]

\[ = 0 \]
The contribution \( \tilde{E}_{12} \) comes from the Hauloul-

\[ H_{\text{Hauloul}} = - \frac{g^2}{2} \left( \Phi_9 + \Phi_g \right) \frac{1}{\tilde{D}^2} \frac{1}{\tilde{D}^2} \left( \phi_9 + \phi_g \right) \]

\[ \tilde{D}^2 = \tilde{D}^2 + \Phi_9 \tilde{D} \tilde{A} x = \tilde{D}^2 \left( 1 + \frac{\tilde{D}^2 \Phi_9 \tilde{D} \tilde{A} x}{\tilde{D}^2} \right) \]

\[ (\tilde{D} D)^{-1} = \left( 1 + \frac{\tilde{D}^2 \Phi_9 \tilde{D} \tilde{A} x}{\tilde{D}^2} \right) (\tilde{D}^2)^{-1} = \left( 1 - \gamma + \gamma' \right)^{-1} (\tilde{D}^2)^{-1} \]

\[ \gamma' = \frac{\tilde{D} \Phi_9 \tilde{D} \tilde{A} x}{\tilde{D}^2} \]

\[ \frac{1}{\tilde{D} \tilde{D}} \left( \frac{1}{\tilde{D}^2} \right) = \left[ 1 - \gamma + \gamma' \right] \left[ 1 - \gamma + \gamma' \right] \]

\[ \left( \frac{1}{\tilde{D}^2} \right) \left( \frac{1}{\tilde{D}^2} \right) = \frac{1}{\tilde{D}^4} \left[ 1 - \gamma + \gamma' \right] \]

\[ = - \left[ 1 - \gamma + \gamma' \right] \frac{1}{\tilde{D}^2} \left[ 1 - \gamma + \gamma' \right] \]

\[ = - \frac{1}{\tilde{D}^2} - \gamma' \frac{1}{\tilde{D}^2} - \gamma' \frac{1}{\tilde{D}^2} - \frac{1}{\tilde{D}^2} \gamma' \]

\[ = - \frac{1}{\tilde{D}^2} - \frac{3 \tilde{D} \Phi_9 \tilde{D} \tilde{A} x}{\tilde{D}^2} \frac{1}{\tilde{D}^2} - \frac{3 \tilde{D} \Phi_9 \tilde{D} \tilde{A} x}{\tilde{D}^2} \frac{1}{\til{D}^2} \frac{1}{\til{D}^2} \]

\[ = - \frac{1}{\til{D}^2} + 3 \til{D}^2 \frac{1}{\til{D}^2} \til{D} \Phi_9 \frac{1}{\til{D}^2} \til{D} \til{A} x \frac{1}{\til{D}^2} \]

\[ = - \frac{1}{\til{D}^2} \til{D}^2 \frac{1}{\til{D}^2} \til{D} \Phi_9 \til{D} \til{A} x \frac{1}{\til{D}^2} \]
Let's choose the integral:

\[ \int \frac{a^2 u}{|u|^3} \left(1 - \cos^2 \theta \right) \, d\theta \quad \text{for } \theta \in (0, \pi) \]

\[
\int \frac{k^2 du \, d\theta}{|u|^3} \left(1 - \cos^2 \theta \right) \quad \text{for } u \in (0, \pi) \]

\[
= \frac{1}{(2\pi)^2} \int k \, dk \int_0^1 dx \frac{(1 - x^2)}{p^2 + u^2 - 2ux}(\frac{1 - x^2}{p^2 + u^2 - 2ux})
\]

We see that as \( u \to \infty \) the integrand behaves like

\[ \frac{k \, dk}{u^2} \ll \frac{du}{u} \quad \text{thus as } u \to \infty \text{ in the upper limit} \]

The integration over \( k \) is going to diverge like \( \ln(\infty) \)

\[ \text{The lower limit } u \to 0 \text{ there is no problem since the integrand goes to } u \to 0 \ll \frac{k(1 - x^2) \, du}{p^2} \to 0 \]

Let's therefore put our UV cutoff. And define

\[ I(\Lambda; p) = \frac{1}{(2\pi)^2} \int k \, dk \int_0^1 dx \frac{(1 - x^2)}{p^2 + u^2 - 2ux} \]

From dimensional reasons we see that \( I \) can only depend on \( \frac{\Lambda}{p} \Rightarrow I = I(\frac{\Lambda}{p}) \)

Instead of calculating \( I \) directly lets calculate in the limit \( \Lambda \to \infty \)

\[ \frac{dI}{d\Lambda} = \left( \frac{\Lambda}{p} \right)^2 \int_0^1 \frac{(1 - x^2)}{p^2 + \Lambda^2 - 2\Lambda p \, n_x} = \frac{1}{(2\pi)^2} \left( \frac{\Lambda}{p} \right)^2 \int_{-1}^{1} \frac{(1 - x^2)}{1 - (\frac{p}{\Lambda})^2 + 2(\frac{p}{\Lambda})} \, dx \]
\[ t = \frac{n}{p} \]
\[ \Lambda \frac{d}{dt} I(\frac{n}{p}) = t \frac{d}{dt} I(t) = \frac{1}{(\pi n)^2} \int_0^t \frac{1}{s^2} \frac{1}{t-s} \, ds = \frac{1}{(\pi n)^2} \frac{3}{2} \]

Thus
\[ t \frac{d}{dt} I(t) = \frac{y}{3} \frac{1}{(\pi n)^2} \quad \text{as } t \to \infty \]

\[ I(t) = I(0) + \frac{y}{3} \frac{1}{(\pi n)^2} \ln t + O(1) \]

This is a constant.

\[ I(t) \sim \ln t + O(1) \]

\[ I(t) = \frac{y}{3} \frac{1}{(\pi n)^2} \ln t + \text{const} + \frac{1}{t} + \ldots \]

leading behavior as \( t \to \infty \)

Finally, for the Fourier transform \( \hat{E}_{1c}(p) \rightarrow E_{1c}^t(p) = \frac{g^4}{(2\pi)^2} \frac{3}{4} \frac{N_c}{2} \frac{4}{3} \ln t \frac{T_B}{T_{AB} T_{BD} T_{BD}} \frac{p^2}{p^2} \)

\[ E_{1c}^t(p) = \left( \frac{N_c}{2} \frac{g^4}{\pi^2} \ln \frac{1}{p} \right) \frac{T_A T_B}{p^2} \]
The contribution to $\varepsilon_{12}^{a,b}$ diagrammatically looks like and we will spare the calculational details (see T.-D. Lee)

and it gives

$$E_{12}^{a,b}(p) = \left(-\frac{1}{12} \frac{N_c}{2} \frac{g^4}{\pi^2} \ln \frac{4}{p} \right) \frac{T_A^a T_{B_1}^b}{p^2},$$

$$= -\frac{1}{12} \delta_{12}.$$

Thus to $O(g^4)$ the $a\bar{b}$ energy is given by (after F.emperature)

$$E_{12}^{a,b} = E_{12}^{a,b}(\nu_{11}) + E_{12}^{a,b}(\nu_{12}) + E_{12}^{a,b}(\nu_{13}) + E_{12}^{a,b}(\nu_{14}) + E_{12}^{a,b}(\nu_{15}) + E_{12}^{a,b}(\nu_{16})$$

$$= 0 \quad 0 \quad 0 \quad 0$$

$$E_{12}^{a,b} = \frac{T_A^a T_{B_1}^b}{4\pi |\vec{r}_1 - \vec{r}_2|^2} g^2 \left[ 1 + \left( \frac{d}{\frac{1}{12}} \right) \right] \text{ since } |\vec{p}|^2 \approx \frac{1}{(\vec{r}_1 - \vec{r}_2)^2}$$

$$= \frac{g^2}{4\pi} \frac{\beta_0 \ln \frac{1}{\vec{r}_1 - \vec{r}_2}}{\vec{r}_1 - \vec{r}_2}$$

define $d = \frac{g^2}{4\pi}$ where $\beta_0 = \frac{11}{3} N_c = 11$ for QCD with $N_c = 3$
\[ E_{ll}(\nu_{l}) = \chi \left[ 1 + \alpha \frac{\beta_0}{4\pi} \ln \frac{\Lambda_i^3}{\Lambda_j^3} \right] \frac{T_{AA}^x T_{BB}^x}{|\nu_i - \nu_j|} \]

What do we do with a dependence, since \( E_{ll}(\nu_{l}) \) is supposed to be a measurable quantity, it better be finite. We will argue that once we introduce a cutoff (\( \Lambda \)) to the theory \( \chi \) which we treated as a constant should only become dependent on \( \Lambda \).

Thus, if we insist that to \( O(\alpha^2) \) this dependence is given by:

\[ \chi(\Lambda) = \chi(\Lambda_0) \left[ 1 - \chi(\Lambda_0) \frac{\beta_0}{4\pi} \ln \frac{\Lambda^2}{\Lambda_0^2} \right] \]  

(\( \chi \))

then:

\[ E_{ll}(\nu_{l}) = \chi(\Lambda_0) \left[ 1 + \chi(\Lambda_0) \frac{\beta_0}{4\pi} \ln \frac{\Lambda^2}{\Lambda_0^2} \ln \frac{\nu_i - \nu_j}{\nu_0} \right] \frac{T_{AA}^x T_{BB}^x}{|\nu_i - \nu_j|} \]

is finite depends on a fixed value of \( \chi(\Lambda + \Lambda_0) \) and good to order \( \alpha^2 \sim g^4 \)
Discussion:

Now \( \alpha(x) \) it follows that \( \alpha(x) \) decreases with \( x \).

If \( |v_i - v_2| \rightarrow 0 \) then we can choose \( \alpha \sim \frac{1}{|v_i - v_2|} \) and therefore be large \( \Rightarrow \alpha(N_0) \) be small.

This means that as \( |v_i - v_2| \rightarrow 0 \) the interaction between quarks becomes weak \( \Rightarrow \) asymptotically it goes to \( 0 \Rightarrow \) asymptotic freedom.

This is due to \( \delta > 0 \) or \( E_{i2} \alpha > 0 \)

the other contribution \( E_{i2} \alpha < 0 \) always (second order pert theory).

But \( \left| E_{i2} \right| = \frac{1}{2} \left| E_{i2} \right| \).

And the possible term wins. Note that both are never thus vanish in QCD.

If we had allowed for light quarks there would also be a negative contribution from the loop of light quarks.
This exists both in QCD and QED and leads to the standard screening of charge i.e.

\[ d \sim (N) \rightarrow \text{increases when } N \text{ increases} \]