Real time Green's function.

So far we looked at thermodynamic properties in equilibrium and now we want to look at exciton-state spectrum and its response to external probing with real time dependence.

We want to introduce G's functions which are both B and real time dependence.

Recall at $T = 0$

$$i \delta_{a\beta}(x,t;x',t') = \langle \psi_0 | T \psi_{aH}^+(x,t) \psi_{\beta}^+(x',t') | \psi_0 \rangle / \sqrt{\nu_0} \nu_0$$

so define:

$$i \delta_{a\beta}(x,t;x',t') = T \langle \psi_{aH}^+(x,t) \psi_{\beta}^+(x',t') | \psi_0 \rangle / \sqrt{\nu_0} \nu_0$$

and given canonical field operator:

$$\hat{\mathcal{E}} = \hat{\mathcal{H}} - \mu \hat{\mathcal{N}}$$
\[ \psi_{\alpha} \] is a Heisenberg spinor operator calculated with \( \kappa \) as the "Hamiltonian."

\[ \psi_{\alpha}(\vec{r}t) = e^{-i\vec{p}\cdot\vec{r}} e^{-i\epsilon t} \text{ unity} \]

\( \Rightarrow \) \( g \) depends on \( \Gamma \) and \( \mu \) \( (\vec{r}t, \vec{p}, \vec{x}t, \vec{p}t) \)

\( \Rightarrow \) we need wave elements of time-ordered operators between all states in the Hilbert space.

\( \Rightarrow \) \[ [\hat{g}, \hat{H}] = 0 \] 2-particle differs from the usual 2-particle by a unitary boost.

\( \Rightarrow \) \( g \) is complicated. We will complete \( \tilde{g} \) by making it to 6-dimensional 6's branch \( (\omega + T = 0) \)

based on properties of Green's function in frequency space.
Lemma representation

(for singlet spin, isotopic system)

\[ \langle \mathbf{G}_{\mu \nu} (x, x', t, t') \rangle = \delta_{\mu \nu} \mathcal{G}(x-x', t-t') \]

if \( t-t' > 0 \) then:

\[ \langle \mathbf{G}^2 (x, t) \rangle = \frac{1}{2 \pi^2} \text{Tr} \left\{ \mathbf{T}_0 \mathbf{P}_{\mathbf{a}_\mathbf{a}_0} \langle \mathbf{G}(x, t) \mathbf{N}^+ \mathbf{a}_\mathbf{a}_0 \mathbf{N} \rangle \right\} \]

where \( \mathbf{N} \) is the number operator and \( \mathbf{P}_{\mathbf{a}_\mathbf{a}_0} \) is the projection operator.

For the present system \( \left[ \mathcal{H}, \mathbf{N} \right] = 0 \), we also have

\[ \left[ \mathbf{N}, \mathbf{P} \right] = 0 \]

and we assume \( \left[ \mathbf{L}, \mathbf{N} \right] = 0 \)

So we can simultaneously diagonalize these operators:

\[ \langle \mathbf{H} | \mathbf{N} | \psi \rangle = \langle \mathbf{H} | \psi \rangle = (\mathbf{E} - \mu \mathbf{N} \mathbf{N}) | \psi \rangle \]

\[ \langle \mathbf{P} | \mathbf{N} | \psi \rangle = \langle \mathbf{P} | \psi \rangle \]

where

\[ \mathbf{N}_x (x, t) = \mathbf{E} \mathbf{E} \mathbf{N}_x (0) \mathbf{E} \mathbf{E} \mathbf{E} \mathbf{N} \mathbf{N} (x, t) \]
\[ G(x,t) = \frac{1}{2\pi} \sum_{\omega} e^{i\omega t} \frac{e^{-i\omega x}}{2i} \left( \frac{e^{-i\omega x}}{2i} + \frac{1}{\omega - (\epsilon_k - \epsilon_n)} \right) \frac{\omega}{\omega - (\epsilon_k - \epsilon_n) + i\epsilon} \]
Discussion:

$G$ is meromorphic in $\nu$, $w$ plane with simple poles at

$$W = \nu_n - \nu_m = E_n - E_m - \mu (N_n - N_m)$$

$$= E_n - E_m - \mu$$

$$\nu_m \in \{m,1/2,0\} \implies N_m = N_m + 1$$

The residues at poles are determined by

$$\left| \langle \psi_m | \psi_n \rangle \right|^2 \text{ and the density function}$$

$$\beta_{(R-\nu_m,\mu)}$$

We have a double sum $(\nu,\mu)$ over states

and $E_n$ and $E_m$ refer to any energy with

various $k$ of particles. Thus we get

poles above and below the real $\nu$ axis

and $\mu$. All
\[ \sum \quad w \quad \text{not analytic in either half plane} \]

For real \( w \) we can separate the real and

\[ G(\xi, \omega) = \frac{1}{2\pi i} e^{-\rho \xi} \sum e^{-\rho \xi} \left( \omega \right)^{\frac{3}{2}} \delta(\xi - \xi_0 - \xi_0) \]

\[ \langle m_{1/2} | \psi \rangle \] square:

\[ \left\{ \begin{array}{c}
\frac{1 + e^{-\rho}}{\rho} \\
\frac{1 - e^{-\rho}}{\rho}
\end{array} \right\} \delta(\omega - \xi_0 - \xi_0) \]

Poincare wall:

\[ \left( \begin{array}{c}
\frac{1 + e^{-\rho}}{\rho} \\
\frac{1 - e^{-\rho}}{\rho}
\end{array} \right) \]

\[ \begin{array}{c}
\text{outside} \\
\text{inside}
\end{array} \]
The temperature limit ($T \to 0$, $\beta \to \infty$)

\[ P(\beta - k \omega) \]

becomes purely from the ground state.

with a mean # of phonons $N_0$ given by

\[ \left( \frac{\beta \omega}{k} \right) N_0 = \mu \quad \text{(Proof later.)} \]

The double sum reduces to:

\[ \kappa_{i} - \lambda_{i} \to \left\{ \begin{array}{ll}
\epsilon_{i} - \epsilon_{0} - \mu & \text{in the first term} \\
-(\epsilon_{i} - \epsilon_{0}) - \mu & \text{in the second term}
\end{array} \right. \]

\[ G(\beta \omega) \to \frac{1}{2s+1} \sum_{\mu} \left\{ \frac{K_{\mu \nu} N_{\mu}(0) N_{\nu}(0)}{\omega + \mu - (\epsilon_{i} - \epsilon_{0}) + i\epsilon} \right\}
\]

\[ + \left\{ \frac{2\pi (\epsilon_{0} - \epsilon_{i})}{(\omega + \mu + (\epsilon_{i} - \epsilon_{0}) - i\epsilon)} \right\} \]

\[ = G(\mu, \omega + \mu) \quad \text{at} \quad T = 0 \]

So \[ G(\mu, \omega - \mu) \bigg|_{T=0} = G(\epsilon, \omega) \quad \text{(T=0)} \]
Special function:

Define:

\[ S(E, w) = \frac{1}{2} \sum_{\nu} e^{-\beta E_{\nu}} \int_{\mathbb{R}} \delta(\omega - (E - E_{\nu} - E_{\mu})) \frac{1}{(1 + e^{-\beta \omega})} (1 - e^{-\beta \omega}) d\omega dE \]

Then:

\[ \tilde{S}(E, w) = \int_{-\infty}^{\infty} \frac{d\omega'}{(2\pi)^{1/2}} S(E, \omega') \left( \frac{1}{\omega - \omega'} - \frac{1 - e^{-\beta \omega}}{1 + e^{-\beta \omega}} \right) d\omega' dE \]

Let's also define advanced and retarded propagators for the homogeneous system:

\[ \tilde{G}^{\alpha, \beta}_{\alpha, \beta}(\mathbf{x}, t) = \pm \Theta(\pm t) \int_0^\infty \left[ \Phi_{\alpha, \beta}(\mathbf{x}, t) \Phi_{\alpha, \beta}(\mathbf{x}, 0) \right] d\tau \]

Here do we assume non-commutative terms at different time.
de a spectral decomposition \( \rho \) of the representation \( \overline{G} \):

\[
\overline{G}^R, A (\tilde{\eta}, w) = \int_{-\infty}^{\infty} dw' \frac{\rho (\tilde{\eta}, w')}{{w - w'}^2 + i\epsilon}
\]

\( \overline{G}^R \) analytic in upper \( \mathbb{C} \) plane

\( \overline{G}^A \) in lower \( \mathbb{C} \) plane.

rewritten:

\[
\overline{G} (\tilde{\eta}, w) = \frac{1}{1 + e^{-\beta w}} \overline{G}^R (\tilde{\eta}, w) + \frac{1}{1 + e^{\beta w}} \overline{G}^A (\tilde{\eta}, w)
\]

Let's define a function of a complex variable \( \tilde{\eta} \):

\[
\Pi (\tilde{\eta}, \tau) \text{ via:}
\]

\[
\Pi (\tilde{\eta}, t) \equiv \int_{-\infty}^{\infty} dw' \frac{\rho (\tilde{\eta}, w')}{\tau - w'}
\]

so that

\[
\overline{G}^R (\tilde{\eta}, w) = \Pi (\tilde{\eta}, w + i \epsilon)
\]

\[
\overline{G}^A (\tilde{\eta}, w) = \Pi (\tilde{\eta}, w - i \epsilon)
\]
Thus given \( g(h,w) \) we can calculate \( P^* \)

\[
\overline{g^2} \overline{g^4} \rightarrow \overline{g}
\]

Furthermore one can prove that \( \overline{g^2,4}(h,w) \propto \frac{1}{w} \) as \( |w| \to \infty \)

(this is important when discussing convergence properties)

The utility of \( g(h,w) \) comes from the following:

We can determine \( g(h,w) \) from temperature

function \( \mathcal{G} \) (we use a different notation to distinguish) \( \mathcal{G} \) is still:

\[
\mathcal{G}(x,t) = -\frac{1}{2}\left[ \psi^* \psi_{kx}(x) \psi_{kx}(x^2) \right]_{t>0}
\]

\[
\text{modified Heisenberg (will reprove)}
\]

and similar for \( t<0 \)
After Lehmann representation \( \Rightarrow \)

\[
G(k, \omega_n) = \int_{-\infty}^{\infty} \frac{df}{\omega} \frac{f(\omega_n)}{\omega - \omega_n} = R(k, \omega_n)
\]

Note \( \omega_n \) discrete

Thus if we calculate \( G \) using Lehmann Feynman rules we can determine \( R(k, \omega_n) \) at a discrete set of points on the imaginary axis. We can then analytically continue to the real axis using spectral representation. This continuation is unique if \( R(\tilde{\omega}, t) \to \frac{1}{t} \) as \( |t| \to \infty \).
In practice we can compute \( f(h_1, w) \) by treating \( w \) as a continuous variable.

\[
\frac{1}{i} \left( G(h_1, w) - G(h_1, w) \right) \left( \frac{1}{x - w - ie} - \frac{1}{x - w + ie} \right)
\]

\[
\int \frac{dw}{2\pi} g(h_1, w)
\]

\( = g(h_1, w) \)

\( \Rightarrow \) now can get \( \frac{\partial^2}{\partial k^2} \) and \( \overline{\partial} \)

Example: noninteracting gas:

\[
G^0(h_1, w) = \frac{1}{\omega_n - (\epsilon_0 - w)}
\]

\[
\Rightarrow g^0(h_1, x) = \frac{1}{i} \left( \frac{1}{x - (\epsilon_0 - w) - ie} - \frac{1}{x - (\epsilon_0 - w) + ie} \right)
\]

\[= 2\pi \delta(x - (\epsilon_0 - w)) \]
\[ G_0^{\omega} (\tilde{v}_i, \omega) = \int_{-\infty}^{\infty} dw \frac{\tilde{s}(\tilde{v}_i, \omega)}{\omega - \omega + i\epsilon} = \frac{1}{\omega - (E^0_{\nu} - \omega) + i\epsilon} \]

So

\[ G_0^{\omega} (\tilde{v}_i, \omega) = \frac{1}{1 + e^{\beta \omega}} \left( \frac{1}{\omega - (E^0_{\nu} - \omega) + i\epsilon} \right) \]

\[ = \frac{1}{\omega - (E^0_{\nu} - \omega) + i\epsilon} + \frac{1}{1 + e^{\beta \omega}} \left( \frac{1}{\omega - (E^0_{\nu} - \omega) + i\epsilon} \right) \]

\[ = \frac{1}{\omega - (E^0_{\nu} - \omega) + i\epsilon} \]

\[ = \frac{1 - n^0_{\nu}}{\omega - (E^0_{\nu} - \omega) + i\epsilon} + \frac{n^0_{\nu}}{\omega - (E^0_{\nu} - \omega) - i\epsilon} \]

\[ \Box \]