\[ L = L(q_a, \dot{q}_a; 3n) \]

In addition, \( L \) is invariant under a group of
distortations whose elements contain certain
ordinary functions of time. ⇒ quotient definition of
gauge symmetry.

Along the possibility of gauge invariance is
not limited to field theories!

Eq. of motion:
\[
\begin{aligned}
\frac{\partial L}{\partial \dot{q}_a} - \frac{d}{dt} \frac{\partial L}{\partial q_a} &= 0 \\
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_a} &= 0 \\
\frac{\partial L}{\partial q_a} &= 0
\end{aligned}
\]

In classical mechanics, the solution can be uniquely
fixed by initial conditions for generalized coordinates
and its time derivatives. This is not the case.
For gauge invariant systems one it's further complicated for quantum systems.

Without gauge invariance \( L(q_a, q_0, \dot{q}_a) \) by itself does not give our answers.

From \( \frac{\partial L}{\partial \dot{q}_a} = 0 \Rightarrow \dot{q}_a = \dot{q}_a(q_a; \dot{q}_a) \)

\( \Rightarrow L = L(q_a, \dot{q}_a; \dot{q}_a(q_a; \dot{q}_a)) = L(q_a; \dot{q}_a) \)

and generalized momenta: \( p_a = \frac{\partial L}{\partial \dot{q}_a} = \frac{\partial L}{\partial \dot{q}_a} \)

\( \Rightarrow \) make quantization problem!

With gauge invariance situation is different.

In general, the dependence of \( \Pi \) will impose functional relation between \( p_a(q_a) \) and \( \dot{q}_a \).

Which makes it impossible to simultaneously have

\( \frac{\partial L}{\partial \dot{q}_a} = 0 \) and the quantization condition:

\[ L p_a(q_a) = \frac{1}{2} \frac{\partial L}{\partial \dot{q}_a} = \frac{1}{2} \alpha \]
This is the underlying reason why we have to choose a specific gauge to carry out quantization.

A simple mechanical model:

Consider a particle in a 3-dimensional space with coordinates:

\[ L = \frac{1}{2} \left( \vec{\tau} - \vec{\alpha} \times \vec{\rho} \right)^2 - V(\rho) \]

\( \vec{\rho} \) is another coordinate vector but \( \vec{\rho} \) is absent in \( L \). This Lagrangian is invariant under

the transformation, \( \vec{\rho} \rightarrow \vec{\rho} + \vec{\alpha} \times \vec{\rho} \)

\( \vec{\rho} \rightarrow \vec{\rho} + \vec{\alpha} \times \vec{\rho} + \vec{\alpha} \)

where \( \vec{\alpha} \) is arbitrary "kinematic" function of time

\( \vec{\alpha} = \vec{\alpha}'(t) \)
Let's simplify the considerations by assuming the system is in \( x_1, x_2 \) plane. And \( \theta = \frac{x}{y} \).

\[
L = \frac{1}{2} (x_1^2 + x_2^2) - (x_1 x_2 - x_2 x_2) \frac{\theta}{2} + \frac{1}{2} \frac{e^2}{r^2} V - V(r)
\]

Here, \( x_1 \) and \( x_2 \) are the Cartesian coordinates.

In polar coordinates:
\[
\begin{align*}
  x_1 &= r \cos \theta \\
  x_2 &= r \sin \theta
\end{align*}
\]

\[
L = \frac{1}{2} \left[ r^2 + r^2 (\dot{\theta} - \dot{x})^2 \right] - V(r)
\]

The symmetry is:
\[
\theta \rightarrow \theta + a(t)
\]

\[
\xi \rightarrow \xi + a(t)
\]

\( a(t) \) can now be our finite function of time.

Eq. of motion:
\[
\frac{\partial L}{\partial \dot{\xi}_n} = 0 = -\tau (\dot{\theta} - \dot{x}) = 0 \quad (+)
\]

Wavenumber connected to \( \theta \) is

\[
\rho = \frac{\partial \rho}{\partial \theta} = \sqrt{2 (\dot{\theta} - \dot{x})} \quad \text{we see that the}
\]

vectors in position \( \text{vec} \{ P_0, \theta \} = -\hat{r} \) is inversely related to \( + \).
The Lorentz equation can be written down without difficulty:

\[ y^2 = \frac{dv}{dr} \quad \Rightarrow \quad \hat{y}^2 = 0 \]

\[ z = 0 \quad \text{gauge} \]

Because of the gauge transformation our orbit:

\[ z = \tilde{z}(t) \quad \Rightarrow \quad \tilde{z} = z(t) \]

can be renumbered to one which has \( z = 0 \) at all times.

\[ L = \frac{1}{2} \hat{v}^2 - V(r) \quad \Rightarrow \quad \hat{p}^2 = \frac{2L}{\hat{v}} = \hat{z}^2 \]

\[ H = \frac{1}{2} \hat{p}^2 + V(r) \quad \text{with} \quad \hat{p}^2 = -i\hat{D} \]

The quantum momentum operator

\[ \hat{p}_\theta = -i \frac{\partial}{\partial \theta} = -i \left[ \frac{1}{i\hbar} \frac{\partial}{\partial \theta} - \hbar \frac{\partial}{\partial \phi} \right] \]

commutes with \( H \). To be consistent...
with the equation to motion:

\[ \nabla^2 (\theta - \phi) = 0 \]

Only eigenvalues \( N \in \mathbb{N} \).

But \( \rho_0>0 \) will be oriented. Hence \( \phi \).

These eigenvalues are all \( \theta \)-independent.

\[ \Rightarrow H = -\frac{1}{2
abla} \left( \frac{\partial}{\partial \nu} (\nabla \frac{\partial}{\partial \nu}) + \nabla \nabla \right) \quad (\star \star \star) \]

Thus in \( z = 0 \) we can zeroize \( \frac{\partial L}{\partial ^2 u} = 0 \)

with \( \Gamma_{\nu}, \phi \) = const. by keeping the quasino

with \( \Gamma_{\nu}, \phi \) = -i \( \Gamma \) + replacing \( \nabla^2 \) be \( \nabla \times \nabla \).
\[ x_2 = 0 \text{ gauze}. \]

Let's choose \( x_2 = 0 \) then \( x_1 \) can be \( >0 \) or \( <0 \)
with \( x_4 = 0 \) being the point when the horizon of the toroidal is two (Guibas ambiguous).

In \( x_2 = 0 \) gauze \( r = x_4 \) and:
\[ L = \frac{1}{2} k_1 k_2 + \frac{1}{2} \xi^2 \dot{x}_1^2 - U(x_4) \]
when we have chosen \( x_4 > 0 \)
\[ \frac{k_3}{\xi^2} = 0 \Rightarrow \frac{x_1^2}{\xi} = 0 \quad \Rightarrow \text{we won't eliminate} \]
\[ p_1 = \frac{\dot{x}_1}{\xi} = x_4 \]
\[ H = \frac{1}{2} p_1^2 + U(x_4) \]

In passing to \( \mathfrak{b} \mathfrak{m} \), in order to the spectrum of this Hamiltonian to be the same as \( (x_1, x_2) \)

It is important not to treat \( x_1 \) in \( \mathfrak{b} \mathfrak{m} \) \( x_1 = 0 \)

gauze is a Coriolis coordinate.
$$p_1 = -\frac{1}{\chi_1} \frac{\partial}{\partial \chi_1} \left( \chi_1 \frac{\partial}{\partial \chi_1} \right)$$

$$\omega_1 - \frac{e^2}{\varepsilon \chi_1}.$$