Now we can discuss what happens under coordinate transformations.

Let's start from a problem in Cartesian basis:

\[ \dot{L} = \frac{1}{2} \dot{w} x^2 \Rightarrow \bar{x} = (x_1, x_2, \ldots, x_n) \]

Under a curvilinear transformation \( x \rightarrow \bar{y} = g(x) \)

We have coordinate change:

\[ L = \frac{1}{2} \bar{x}^T \bar{X} - V \text{ becomes:} \]

\[ L(\bar{y}, \dot{\bar{y}}) = \frac{1}{2} \dot{\bar{y}}^T M(\bar{y}) \dot{\bar{y}} - V(\bar{y}) \]

where \( M = M_{ab} = \frac{\partial x_c}{\partial \bar{y}^a} \frac{\partial x_c}{\partial \bar{y}^b} \)

The classical Hamiltonian:

\[ p_a = \frac{\partial \mathcal{L}}{\partial \dot{\bar{y}}^a} = (M \cdot \dot{\bar{y}})_a = M_{ab} \dot{\bar{y}}^b \]

\[ \Rightarrow \dot{\bar{y}} = M^{-1} p \]
The time-dependent Schrödinger equation is

$$ H \Psi(t) = i \frac{\partial}{\partial t} \Psi(t). $$

Stationary states: $H |a\rangle = E_a |a\rangle$

$$ \Psi_a(x,t) = \langle x | a \rangle $$

$\Psi_a(x)$ forms a complete set of states:

$$ \int d^Dx \Psi_a(x) \Psi_a(x) = \delta_{aa}. $$

In curvilinear coordinates $d^Dx = \int d^Dq$

$$ d^Dx = \int d^Dq |x\rangle \langle x| $$

$$ |x\rangle = \int d^Dq |q\rangle \langle q| $$

$$ \Rightarrow \phi_a(q,t) = \Psi_a(x(q),t) \cdot \sqrt{f(q)} $$

$$ \phi_b(q) = \Psi_a(x(q)) \cdot \sqrt{f(q)} $$

$$ \Rightarrow \int d^Dq \phi^\dagger_a(q,t) \phi_b(q,t) = \int d^Dx \Psi^\dagger_a(x(t)) \Psi_b(x) = \delta_{ab}. $$
\[ H(p, q) = p^T \dot{q} - L = p^T M^{-1} \dot{p} - \frac{1}{2} p^T M^{-1} H M^{-1} \dot{p} + V(q) \]

\[ H_a = \frac{1}{2} p^T M^{-1}(q) \dot{p} + V(q) \]

The quantum Hamiltonian in the canonical coordinates is

\[ H = -\frac{1}{2} \nabla^2 + V(x) \]

\[ \vec{\nabla} \equiv \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right) \]

The same Hamiltonian in the curvilinear coordinates is (see homework)

\[ H(p, q) = \frac{i}{\hbar} \sum_{\alpha} \left( \hbar \partial_{q_\alpha} - i \hbar \omega_{q_\alpha} \right) a_\alpha \left( \hbar \partial_{p_\alpha} - i \hbar \omega_{p_\alpha} \right) a_\alpha^\dagger + V(q) \]

where \( p_\alpha = -i \hbar \partial_{q_\alpha} \) and \( \hbar = \sqrt{\text{det} M(q)} \)
Also define \( \overline{H} = \sqrt{J(q) \, H(p, q) \, \frac{1}{\sqrt{J(q)}}} \)

operator.

Here:
\[
\overline{H} \, \phi_a(q) = E_a \, \phi_a(q)
\]

\[
\overline{H} \, \phi(q, t) = i \frac{\partial}{\partial t} \phi(q, t)
\]

and
\[
\langle q' t' | q t \rangle = \langle q' | e^{-i (t'-t) \overline{H}} | q \rangle = \sum_a \phi_a(q') \phi_a^*(q) e^{i E_a (t'-t)}
\]

Thus:
\[
\phi(q', t') = \int dq \, \langle q' t' | q t \rangle \phi(q, t)
\]

Therein:
\[
\langle q' t' | q t \rangle = \text{Det} \int A \left( \prod_{i=1}^{N-1} \int d^D q_i \, e^{i \sum_{i=0}^{\text{Left- } n} T_i} \right)
\]

where \( \text{Left- } = \sum_{n} \left( g(q(n)) - \frac{\nu}{\varepsilon} \sqrt{g(q(n))} \right) - V_c(q_n) \)

with \( \sqrt{A} = (\text{Det})^{\frac{1}{2L}} \)
When as usual \( q_n = \frac{9n + 9n - 1}{2} \), \( q_n = \frac{9n + 1 - 9n}{2} \).

And \( V_c(q) = \frac{1}{8} \left[ \delta(qa)(\frac{3}{xc}) \right] \left[ \delta(qb)(\frac{1}{xc}) \right] \).

The Jacobian term becomes an introducing "effective potential" with a magnitude proportional to \( \epsilon \).

\( \Rightarrow \) as \( \epsilon \to 0 \) there is also an additional real potential \( V_c \).

Proof: The Hamiltonian operator corresponding to

\( H = \frac{1}{2} p^2 + V(q) \) is:

\( H(p,q) = \frac{1}{2} \left[ \frac{p_a p_b M_{ab}}{M_{ab}} + 2 p_a M_{ab} p_b + M_{ab} p_a p_b \right] + V(q) \)

with \( p_0 = -i \frac{\partial}{\partial q_0} \)

and \( M_{ab} = \frac{\partial q_a}{\partial x} \frac{\partial q_b}{\partial x} \).
By straightforward differentiation one can show that the difference between $\overline{H}$ and $H$ is:

$$
\overline{H}(p, q) = H(p, q) + V_c(q)
$$

so:

$$
\langle q | e^{-i(t-t') \overline{H}} | q \rangle = \mu_n \Gamma \frac{1}{\sqrt{(2\pi)^d}} e^{-iE_n} e^{i\langle q, q \rangle}
$$

g as $n \to 0$

and:

$$
\Sigma_n = p_n \cdot \dot{q}_n - H_c \left( p_n \cdot \dot{q}_n \right) - V_c(q_n)
$$

The function $\Sigma_n$ depends quadratically on $p_n$.

Now $H$ is real and symmetric. Here is a real and orthogonal matrix $U$ which diagonalizes $H$.

$$
U^T H(q_n) U = \Lambda = \left( \begin{array}{cccc}
\lambda_1 & 0 & \cdots & 0 \\
0 & \ddots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \lambda_d
\end{array} \right)
$$
Define new D-dimensional vectors (at each time step):

\[ \tilde{v}_n = U^T \hat{p}_n \] and \[ v_n = U^T q_n \]

So that:

\[ \tilde{v}_n = \left( \tilde{J}^T v - \frac{1}{2} \tilde{J}^T \Lambda \tilde{J} - v - v_c \right)_n \]
\[ = -\frac{1}{2} \sum_{\alpha = 1}^{D} \left[ \frac{1}{\alpha} \left( \tilde{J}_\alpha \left( \tilde{J}_\alpha \right) - \tilde{J}_\alpha \tilde{J}_\alpha \right) \right] - v - v_c \]

Here:

\[ d^{D} p_n = d\tilde{p}_n(n) - d\tilde{p}_0(n) \]

and \[ \sum_{\alpha} \tilde{J}_\alpha \tilde{J}_\alpha \] = \[ v^T \Lambda \Lambda = q_n^T M(q_n) q_n \]

and \[ \prod_{\alpha = 1}^{D} \tilde{\lambda}_\alpha = \sqrt{\det M(q_n)} = J(q_n) \]

Therefore, the integration can be performed as follows:

a Jacobian at each time step.
\[ \langle q', t' | q, t \rangle = \langle q', \frac{1}{e^t} | q \rangle \]
\[ = \int D(q) \ e^{-it' [L_{eff}(q) - \frac{i\pi\hbar}{\nu} - \sum_{e=0}^{E} \nu_e(q)]} \]

Example:

As dynamical systems both QED and QCD belong to a class of Lagrangians known whose generalized coordinates can be separated into two types \( q_a \)'s and \( \xi_a \)'s. The Lagrangian is a function of \( q_a, \dot{q}_a, \dot{\xi}_a \) but not \( \ddot{\xi}_a \).