Problem set 1
Due on or before February 1st
For reference use lecture notes and F&W.

1. (10pt) Let \( P_m \) be the probability for a system to be in a state \( m \), and \( V_{mn} = \langle m | V | n \rangle \). Time-dependent perturbation theory gives

\[
\frac{dP_m}{dt} = \sum_n \left( |V_{mn}|^2 P_n - |V_{mn}|^2 P_m \right) \tag{1}
\]

Use this result to show that entropy increases with time.

2. Consider an ideal relativistic Fermi gas described by the Hamiltonian

\[
H_0 = \sum_{k, \lambda} e(k) \left( b^\dagger(k, \lambda)b(k, \lambda) + d^\dagger(k, \lambda)d(k, \lambda) \right) \tag{2}
\]

where \( e(k) = \sqrt{k^2 + m^2} \). This is the same Hamiltonian you studied in the last homework in P625,

\[
H_0 = \int d^{\lambda} \psi^\dagger(x) [-i\alpha \cdot \nabla + \beta m] \psi(x) \tag{3}
\]

once you write the field operators as

\[
\psi(x) = \sum_{k, \lambda} \left[ u(k, \lambda)b(k, \lambda) + \nu(-k, \lambda)d(-k, \lambda) \right] \frac{1}{\sqrt{V}} e^{ik \cdot x} \tag{4}
\]

and similarly for \( \psi^\dagger \). Here \( b \) and \( d \) operators annihilate particles (quarks) and antiparticles (antiquarks) respectively, and similarly for the creation operators.

(5pt) Show that the number operator \( N = \int d^{\lambda} \psi^\dagger \psi \) is given by

\[
N = \sum_{k, \lambda} b^\dagger(k, \lambda)b(k, \lambda) - d^\dagger(k, \lambda)d(k, \lambda) \tag{5}
\]

and that is commutes with \( H_0 \).

10 pt Compute the noninteracting grand partition function \( Z_G^0 \) and the grand potential \( \Omega_0 \). Determine the quark and antiquark occupation numbers as functions of \( k, T, \mu \) and derive equation which enable to solve for \( \mu \) in terms of the fermion number density \( N/V \).

10 pt Use thermodynamical relations to compute (reduce to quadrature) energy \( E \) as function of \( T, V, \mu \), and determine the equation of state \( (pV \text{ as function} \)
of $E$) in the limit of massless quarks. Discuss the physics situation that occurs when $\mu = 0$ and $T$ is finite as it was in the early Universe.

10pt Restore the quark masses and discuss the $T \to 0$ limit. Determine the value of $\mu/k_B T$ in the limit $T \to \infty$ and compare with that for the nonrelativistic gas. Show that the Stefan-Boltzman law is reproduced what $T \to \infty$ and determine the equation of state.