Problem set 3
Due on or before Wednesday, March 20
For reference use lecture notes, Walecka’s book and Fetter and Walecka’s book

1 In this problem you will do numerical computation of a path integral. But first some analytical calculations.

(10pt) Consider a single particle with mass $m = 1$ in one space dimension. Define

$$K_{b,a} \equiv \langle x_b, \tau | x_a, \tau \rangle \equiv \langle x_b | e^{-H(\tau) - \tau} | x_a \rangle$$

which is the kernel (transition amplitude) in imaginary time. Prove that

$$\langle x_b, \tau | x_a, \tau \rangle = \int_{x_a(\tau)}^{x_b(\tau)} D[x(t)] e^{-S_E[x(\tau), x_a(\tau)]}$$

where $S_E$ is the Euclidean, action given by,

$$S_E = \int_{\tau_a}^{\tau_b} dt \left[ \frac{1}{2} \left( \frac{dx(\tau)}{d\tau} \right)^2 + V(x) \right]$$

(10pt) Show how from $\langle x_b, \tau | x_a, 0 \rangle$ one can obtain the ground state energy, $E_0$ of the Hamiltonian,

$$H|n\rangle = \left( \frac{\hat{p}^2}{2} + V(\hat{x}) \right)|n\rangle = E_n|n\rangle$$

(40pt) Now numerics. Consider a harmonic oscillator potential $V(x) = x^2/2$ ($\omega = 1$). Once the path integral is discretized by dividing $\tau = \tau_b - \tau_a$ into $N$ time slices,

$$N\epsilon = \tau$$

it depends on $x_a, x_b, N, \epsilon$. Compute the kernel numerically for large $N$, say $N = 10$. The best strategy is to use Monte Carlo techniques. You can either write a simple MC generator to evaluate the $N - 1$ dimensional integral or use one of existing numerical packages (e.g. vegas by P.Lepage). Show how the ground state is obtained. For this you should draw a plot of $E_{MC} = E_{MC}(?)$ where $?$ is the relevant variable (you need to come up with one) and demonstrate how your numerical results converge, $E_{MC} \rightarrow E_0$ in the continuum limit.

2. When discussing path integrals, in particular in nonabelian field theory, one often needs to consider nonlinear coordinate transformations. Consider a
system with \( N \) degrees of freedom described by the generalized coordinates, \( x_i, \ i = 1, \cdots N \), and a general coordinate transformation to the coordinates, \( x_i \to q_j = q_j(x_i) \) Define \( M \) to be the \( n \times n \) matrix, (repeated indices are summed over)

\[
M = M_{ij} = \left( \frac{\partial x_k}{\partial q_i} \frac{\partial x_k}{\partial q_j} \right)
\]

(6)

\( M^{-1} \) to be its inverse and \( |M| \equiv J^2 \) its determinant.

Show that (10pt)

\[
\frac{1}{|M|} \frac{\partial |M|}{\partial q_k} = M^{-1}_{ji} \frac{\partial M_{ij}}{\partial q_k} = -2 \frac{\partial x_i}{\partial q_k} \frac{\partial q_j}{\partial q_k} \frac{\partial q_j}{\partial x_i}
\]

(7)

(10pt) and

\[
\frac{\partial^2}{\partial x_i \partial x_i} = \frac{1}{J} \frac{\partial}{\partial q_i} \left( M^{-1}_{ij} J \frac{\partial}{\partial q_j} \right)
\]

(8)